Lecture 36: Introduction to Root Locus Concept I

Course Information

- **Course**: Control Instrumentation System (DI03000181)
- **Unit 4**: Concept of Stability
- **Topic**: 4.4 Introduction to Root Locus Concept

Learning Objectives

- Understand the fundamental philosophy of W. R. Evans' Root Locus method.
- Formulate the closed-loop characteristic equation and derive the Angle and Magnitude conditions.
- Identify how root trajectories in the complex s-plane govern dynamic stability and transient performance.
- Apply foundational construction rules to sketch qualitative root loci for control instrumentation systems.

Lecture Agenda I

1. Motivation and Historical Perspective of Root Locus
2. Closed-Loop Characteristic Equation and Parameter Variation
3. Derivation of Angle and Magnitude Conditions
4. Vector Evaluation of $G(s)H(s)$ on the s-Plane
5. Core Construction Rules (Symmetry, Branches, Real Axis)
6. Asymptotes, Centroid, and Breakaway Behavior
7. Comprehensive Summary Table of Root Locus Properties
8. Illustrative Case Study: 3rd-Order System Root Locus
9. Applications in Control Instrumentation & Summary

Motivation and Concept of Root Locus I

- **The Core Challenge**: Closed-loop poles determine system response features (damping ratio ζ, natural frequency ω_n, settling time T_s, and stability). However, varying controller gain K continuously alters the polynomial coefficients of the closed-loop characteristic equation.
- **Definition of Root Locus**: A graphical plot in the complex s-plane showing the trajectory of closed-loop poles as a single system parameter (typically open-loop gain K) varies from 0 to $+\infty$.
- **Why Open-Loop Information Matters**: Finding roots of high-order characteristic equations for every K is computationally tedious. Walter R. Evans (1948) developed a graphical method to map open-loop poles and zeros directly to closed-loop pole paths.

Key Advantage

Root locus allows control engineers to predict stability boundaries and transient performance variations without explicitly solving high-degree characteristic polynomials.

Characteristic Equation and Gain Variation I

Closed-Loop System Dynamics

Consider a standard negative feedback control loop with open-loop transfer function $G(s)H(s)$:

$$T(s) = \frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Characteristic Polynomial

The closed-loop poles are the roots of the characteristic equation:

$$1 + KG(s)H(s) = 0 \implies KG(s)H(s) = -1$$

Expressing $G(s)H(s)$ in factored pole-zero format (n poles, m zeros with $n \geq m$):

$$G(s)H(s) = \frac{\prod_{j=1}^{m}(s + z_j)}{\prod_{i=1}^{n}(s + p_i)}$$

Characteristic Equation and Gain Variation II

- **Limit as** $K \to 0$**:** $KG(s)H(s) \to 0 \implies \prod_{i=1}^{n}(s+p_i) \to 0$. The locus starts at the **open-loop poles**.
- **Limit as** $K \to \infty$**:** $G(s)H(s) \to 0 \implies \prod_{j=1}^{m}(s+z_j) \to 0$. m branches terminate at **open-loop zeros**, and $(n-m)$ branches terminate at **infinity**.

Angle and Magnitude Conditions I

Since $KG(s)H(s) = -1$ is a complex equation, it resolves into two fundamental scalar requirements for any point $s = s_0$ on the locus:

1. Angle Condition (Existence Criterion)

$$\angle G(s_0)H(s_0) = \pm(2k+1)180^\circ, \quad k = 0, 1, 2, \ldots$$

Summing component vectors from all open-loop zeros and poles to point s_0:

$$\sum_{j=1}^{m} \angle(s_0 + z_j) - \sum_{i=1}^{n} \angle(s_0 + p_i) = \pm(2k+1)180^\circ$$

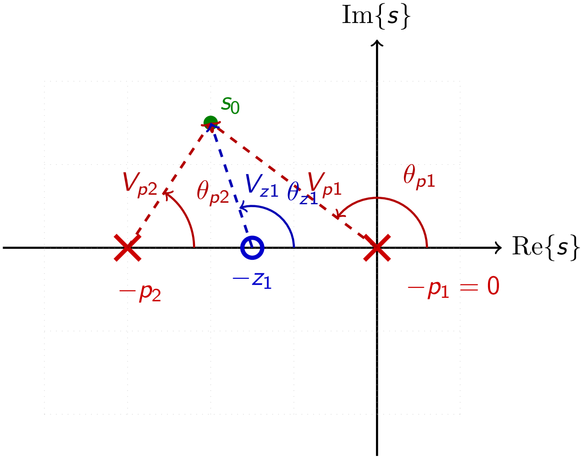
Note: The angle condition solely determines whether a point s_0 lies on the root locus.

2. Magnitude Condition (Gain Calibration)

$$|KG(s_0)H(s_0)| = 1 \implies K = \frac{1}{|G(s_0)H(s_0)|} = \frac{\prod_{i=1}^{n} |s_0 + p_i|}{\prod_{j=1}^{m} |s_0 + z_j|}$$

Note: The magnitude condition is used to calculate the specific value of $K > 0$ at a point s_0 known to be on the locus.

Vector Evaluation of Angle Condition I



Angle Condition: $\sum \theta_z - \sum \theta_p = \theta_{z1} - (\theta_{p1} + \theta_{p2}) = \pm 180^\circ (2k+1)$

Construction Rules: Real-Axis Segments & Symmetry I

Rule 1: Symmetry of Locus

The root locus is always symmetrical with respect to the real axis ($\mathrm{Re}\{s\}$). Complex roots of real-coefficient characteristic polynomials must occur in conjugate pairs.

Rule 2: Number of Locus Branches

The number of separate branches of the root locus equals the number of open-loop poles n (assuming $n \geq m$, which is standard for physically realizable causal systems).

Construction Rules: Real-Axis Segments & Symmetry II

Rule 3: Real Axis Existence Rule

A point on the real axis lies on the root locus if and only if the total number of real open-loop poles and real open-loop zeros to its right is **ODD**.

- Poles/zeros to the left of the point contribute 0° angle.
- Complex conjugate pole/zero pairs contribute a combined 360° angle.
- Each real pole/zero to the right contributes 180° angle.

Construction Rules: Asymptotes and Centroid I

When $n > m$, $(n-m)$ locus branches move towards infinity. These branches approach straight-line asymptotes.

Asymptote Angles (ϕ_k)

The angles formed by the asymptotes with the positive real axis are given by:

$$\phi_k = \frac{\pm(2k+1)180^\circ}{n-m}, \quad k = 0, 1, 2, \ldots, (n-m-1)$$

Asymptote Centroid (σ_A)

All asymptotes intersect the real axis at a single point called the **centroid** σ_A:

$$\sigma_A = \frac{\sum \text{Real parts of Poles} - \sum \text{Real parts of Zeros}}{n-m} = \frac{\sum_{i=1}^{n}(-p_i) - \sum_{j=1}^{m}(-z_j)}{n-m}$$

Breakaway Points and $j\omega$-Axis Crossing I

Breakaway and Break-in Points

- **Breakaway Point**: Point on real axis where two or more loci moving towards each other coalesce and depart into the complex plane as K increases.
- **Break-in Point**: Point on real axis where complex loci enter the real axis as K increases.
- **Mathematical Determination**: Solve $\frac{dK}{ds} = 0$ for $K = -\frac{1}{G(s)H(s)}$, or solve:

$$\sum_{i=1}^{n} \frac{1}{s + p_i} = \sum_{j=1}^{m} \frac{1}{s + z_j}$$

Imaginary Axis Crossing (Stability Boundary)

The points where the root locus intersects the imaginary axis ($s = \pm j\omega_{cr}$) define the critical gain K_{cr} for marginal stability.

- Can be determined by setting $s = j\omega$ in $1 + KG(s)H(s) = 0$.
- Alternatively determined using the Routh-Hurwitz stability array.

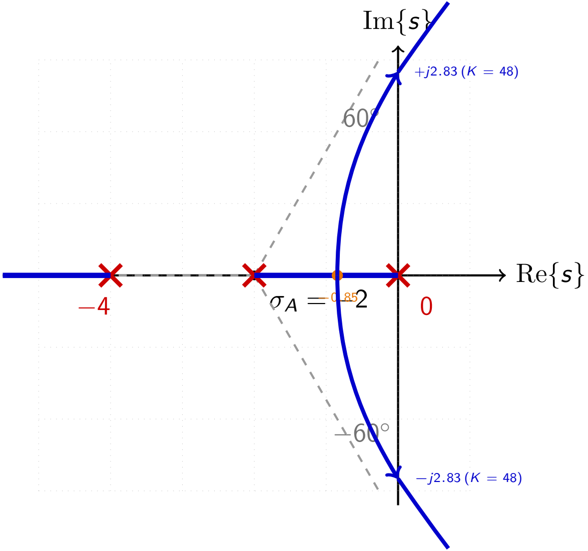
Summary of Root Locus Rules and Parameters I

Table: Key Root Locus Construction Rules and Formulas

Property / Feature	Mathematical Formula	Physical Significance
Starting Points ($K = 0$)	$s = -p_i$ (Open-loop poles)	System open-loop dynamics
Ending Points ($K \to \infty$)	$s = -z_j$ or ∞	Zero locations and asymptotic limits
Number of Branches	$N = n$	Total closed-loop roots
Real Axis Segment	$\sum$(Poles + Zeros to right) = Odd	Region on real axis satisfying angle cond.
Asymptote Angles	$\phi_k = \frac{\pm(2k+1)180^\circ}{n-m}$	Direction of branches going to ∞
Asymptote Centroid	$\sigma_A = \frac{\sum(-p_i)-\sum(-z_j)}{n-m}$	Real axis intersection of asymptotes
Breakaway Condition	$\frac{dK}{ds} = 0$ or $\sum \frac{1}{s+p_i} = \sum \frac{1}{s+z_j}$	Transition from real to complex roots
$j\omega$-Axis Crossing	Routh array row of zeros / $s = j\omega$	Stability limit ($K = K_{cr}, \omega = \omega_{cr}$)

Case Study: Root Locus of a 3rd-Order System I

Case Study: Root Locus of a 3rd-Order System II



Root Locus for $G(s)H(s)$ K showing Centroid, Asymptotes, and $j\omega$ Crossings

Summary and Instrumentation Applications I

Applications in Control Instrumentation

- **Controller Gain Tuning**: Allows engineers to select proportional gain K for target damping ratio ζ and natural frequency ω_n.
- **Stability Margin Analysis**: Directly shows critical gain K_{cr} beyond which closed-loop poles enter the right-half s-plane (instability).
- **Compensator Design**: Guides placement of lead/lag compensator poles and zeros to reshape the locus into desirable s-plane regions.

Key Takeaways

- Root locus visually depicts closed-loop pole trajectories as gain K ranges from 0 to ∞.
- The **Angle Condition** governs locus existence; the **Magnitude Condition** determines parameter values.
- Key geometric markers (centroid, asymptotes, breakaway points) enable fast, accurate locus sketching.

Summary and Instrumentation Applications II

Next Lecture Preview

In **Lecture 37**, we will dive deeper into *Detailed Construction Rules of Root Locus*, focusing on angles of departure/arrival for complex poles and advanced numerical examples.