Lecture 39: Tank circuit

Oscillators

DI02011011: Electronics Circuit and Application (ECA)

Agenda: Tank Circuit Fundamentals

- Operational Principles & Flywheel Energy Exchange in LC Tanks
- Derivation of Resonant Frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$
- Impedance & Admittance Analysis of Ideal vs Practical Tank Circuits
- Quality Factor (Q) Definition, Derivation, and Selectivity
- Differential Analysis of Damped Oscillations & Logarithmic Decrement
- Quantitative Numerical Example: Design & Parameter Calculation
- Tapped Tank Circuits: Hartley Inductive and Colpitts Capacitive Dividers
- Electronic Frequency Tuning via Varactor Diodes and Ferrite Cores
- Loading Effects and Loaded Quality Factor (Q_L)
- Quartz Crystal as an Ultra-High Q Equivalent LC Tank Circuit
- Summary & Key Engineering Takeaways

Operational Principles & Flywheel Energy Exchange

- Energy Storage Elements: Capacitor stores electrostatic energy $E_C = \frac{1}{2}CV^2$; Inductor stores magnetic energy $E_L = \frac{1}{2}LI^2$.
- Flywheel Effect Cycle 1: Fully charged capacitor discharges current $i(t)$ into inductor, building up magnetic flux in inductor coil.
- Flywheel Effect Cycle 2: As capacitor voltage drops to zero, collapsing magnetic field induces electromotive force (EMF) $e = -L\frac{di}{dt}$, recharging capacitor with opposite polarity.
- Self-Sustained Sinusoidal Exchange: In ideal lossless tank $(R_L = 0)$, continuous back-and-forth charge circulation maintains constant amplitude sinusoidal voltage waveform.

Derivation of Resonant Frequency f_0

- Admittance Formulation: Inductive admittance $Y_L = \frac{1}{j\omega L} = -j\frac{1}{\omega L}$; Capacitive admittance $Y_C = j\omega C$.
- Total Parallel Admittance: $Y(j\omega) = Y_L + Y_C = j\left(\omega C - \frac{1}{\omega L}\right)$.
- Resonance Condition: Imaginary part of admittance vanishes: $\mathrm{Im}(Y) = 0 \implies \omega_0 C - \frac{1}{\omega_0 L} = 0$.
- Resonant Frequency Formulas: $\omega_0^2 = \frac{1}{LC} \implies \omega_0 = \frac{1}{\sqrt{LC}} \implies f_0 = \frac{1}{2\pi\sqrt{LC}}$.

Impedance Analysis of Ideal vs Practical Tank Circuit

- Ideal Parallel Tank Impedance: At resonance, $Y(\omega_0) = 0 \implies Z_0 = \frac{1}{Y} \to \infty$ (infinite dynamic impedance, acts as open circuit).
- Practical Coil Resistance: Inductor winding has internal resistance R_L. Inductive branch impedance $Z_L = R_L + j\omega L$.
- Practical Parallel Admittance Derivation: $Y = j\omega C + \frac{1}{R_L + j\omega L} = \frac{R_L + j(\omega L - \omega C R_L^2 - \omega^3 L^2 C)}{R_L^2 + \omega^2 L^2}$.
- Equivalent Parallel Resistance (R_p): At resonance $\omega_0 \approx \frac{1}{\sqrt{LC}}$, real conductance $G_p = \frac{R_L}{\omega_0^2 L^2} \implies R_p = \frac{L}{R_L C} = Q^2 R_L$.

Quality Factor (Q) & Selectivity

- Quality Factor Definition: $Q = 2\pi \frac{\text{Maximum Energy Stored}}{\text{Energy Dissipated per Cycle}} = \frac{\omega_0 L}{R_L} = \frac{1}{\omega_0 C R_L}$.
- Parallel Resistance Form: Expressed in terms of equivalent parallel impedance: $Q = \frac{R_p}{\omega_0 L} = R_p \omega_0 C = R_p \sqrt{\frac{C}{L}}$.
- 3dB Bandwidth Relation: $BW = f_2 - f_1 = \frac{f_0}{Q}$.
- Selectivity & Phase Slope: Higher Q produces narrower 3dB bandwidth, sharper frequency selection, and steeper phase transition slope $\left.\frac{d\phi}{d\omega}\right|_{\omega_0} = \frac{2Q}{\omega_0}$.

Differential Analysis of Damped Oscillations

- Differential Equation: Applying KVL around practical RLC tank: $L\frac{d^2q}{dt^2} + R_L\frac{dq}{dt} + \frac{q}{C} = 0$.
- Damping Factor & Damped Natural Frequency: Damping coefficient $\alpha = \frac{R_L}{2L}$; damped resonant frequency $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$.
- Transient Voltage Response: Under-damped response ($Q > 0.5$): $v(t) = V_0 e^{-\alpha t}\cos(\omega_d t)$.
- Logarithmic Decrement (δ): Natural logarithm of ratio of successive peak amplitudes: \{}delta = \{}ln\{}left(\{}frac{V_n}{V_{n+1}}\{}right) = \{}alpha T_d \{}approx \{}frac{\{}pi}{Q}.

Quantitative Numerical Example: Design & Parameter Calculation

- Problem Statement: An LC tank circuit uses inductor $L = 100\ \mu$H with winding resistance $R_L = 5\ \Omega$, and tuning capacitor $C = 200$ pF. Calculate f_0, Q, R_p, and BW.
- 1. Resonant Frequency: $f_0 = \frac{1}{2\pi\sqrt{100\times10^{-6}\times200\times10^{-12}}} = \frac{1}{2\pi\sqrt{2\times10^{-14}}} = 1.1254$ MHz.
- 2. Reactance & Quality Factor: $X_L = 2\pi f_0 L = 2\pi(1.1254 \times 10^6)(10^{-4}) = 707.1\ \Omega \implies Q = \frac{707.1}{5} = 141.42$.
- 3. Equivalent Parallel Impedance: $R_p = Q^2 R_L = (141.42)^2 \times 5 = 100\ \text{k}\Omega$ (or $R_p = \frac{L}{R_L C} = \frac{10^{-4}}{5\times2\times10^{-10}} = 100\ \text{k}\Omega$).
- 4. 3dB Bandwidth: $BW = \frac{f_0}{Q} = \frac{1.1254\times10^6\ \text{Hz}}{141.42} = 7.958$ kHz.

Tapped Tank Circuits: Inductive & Capacitive Dividers

- Hartley Inductive Tap: Tank inductor split into two series coils L_1 and L_2 with mutual inductance M. Voltage tap fraction $\beta = \frac{L_1+M}{L_1+L_2+2M}$.
- Colpitts Capacitive Tap: Tank capacitor split into series pair C_1 and C_2. Voltage tap fraction $\beta = \frac{V_f}{V_{out}} = \frac{C_1}{C_2}$.
- Impedance Transformation Ratio: Tapping acts as step-down transformer ($n = \frac{N_1}{N_2}$ or $\frac{C_2}{C_1}$), matching high tank impedance R_p (100 kΩ) to low transistor input impedance (1 kΩ).
- Preservation of Tank Q: Tapped impedance transformation prevents heavy loading of main tank resonator by low transistor input impedance.

Electronic Frequency Tuning & Variable Tank Circuits

- Varactor Diode Tuning: Reverse-biased PN junction capacitance $C_v(V_R) = \frac{C_0}{(1+V_R/V_{bi})^n}$. Varying DC tuning voltage V_R sweeps resonant frequency $f_0(V_R) = \frac{1}{2\pi\sqrt{LC_v(V_R)}}$.
- Permeability Tuning: Mechanically shifting high-permeability ferrite core inside inductor coil alters effective inductance $L = \mu_r L_0$, adjusting f_0.
- Switched Capacitor Banks: Digitally switched PIN diode / MOSFET capacitor arrays for wideband multi-band RF synthesizers.
- Air Trimmer Capacitors: Precision mechanical tuning capacitors $(5 - 30 \text{ pF})$ for high-stability frequency calibration.

Loading Effects and Loaded Quality Factor (Q_L)

- Unloaded vs Loaded Tank: Unloaded tank impedance is $R_p = Q_{unloaded}\omega_0 L$. Connecting external load R_{load} places R_{load} in parallel with R_p.
- Total Parallel Load Resistance: $R_{total} = R_p || R_{load} = \frac{R_p R_{load}}{R_p + R_{load}}$.
- Loaded Quality Factor Formula: $Q_L = \frac{R_{total}}{\omega_0 L} = Q_{unloaded} \cdot \left(\frac{R_{load}}{R_p + R_{load}} \right) = \frac{Q_{unloaded}}{1 + R_p / R_{load}}$.
- Impact on Performance: Heavy loading ($R_{load} < R_p$) drastically reduces Q_L, widening bandwidth $BW_L = \frac{f_0}{Q_L}$ and degrading phase noise.

Quartz Crystal as an Equivalent Ultra-High Q LC Tank

- Piezoelectric Quartz Wafer: Electromechanical coupling converts mechanical vibrations into high-precision electrical resonance.
- Butterworth-Van Dyke (BVD) Model: Motional branch with series L_m (Henries), C_m (femtoFarads), R_m (Ohms), in parallel with static electrode capacitance C_0 (picoFarads).
- Dual Resonance Characteristics: Series resonant frequency $f_s = \frac{1}{2\pi\sqrt{L_m C_m}}$; Parallel resonant frequency $f_p = \frac{1}{2\pi\sqrt{L_m \frac{C_m C_0}{C_m + C_0}}}$.
- Ultra-High Quality Factor: Motional inductance L_m is large $(mH$ to $H)$ while motional resistance R_m is small $(10 - 50\ \Omega)$, achieving $Q = \frac{\omega_0 L_m}{R_m} \sim 10,000$ to $500,000$.

Summary: Key Takeaways on LC Tank Circuits

- Resonance Mechanism: Parallel LC tank provides energy exchange between C and L with resonant frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$.
- Impedance & Loss: Practical tank has finite equivalent parallel resistance $R_p = \frac{L}{R_L C} = Q^2 R_L$ at resonance.
- Quality Factor Impact: Quality factor $Q = \frac{\omega_0 L}{R_L}$ controls selectivity, loaded bandwidth $BW = \frac{f_0}{Q_L}$, and phase slope.
- Advanced Configurations: Tapped tanks (Hartley/Colpitts) provide impedance transformation; varactor diodes enable electronic tuning; quartz crystals provide ultra-high Q resonance.