Lecture 37: Barkhausen's criteria for oscillation

Oscillators

DI02011011: Electronics Circuit and Application (ECA)

Agenda: Barkhausen's Criteria for Oscillation

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- Complex S-Plane Analysis & Pole Trajectories
- Startup Criterion ($|A\beta| > 1$) vs Steady-State Criterion ($|A\beta| = 1$)
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Mathematical Statement of Barkhausen's Criteria

- Loop Gain Definition: $T(s) = A(s)\beta(s)$, where $A(s)$ is amplifier open-loop transfer function and $\beta(s)$ is feedback transfer factor.
- Magnitude Condition: $|T(j\omega_0)| = |A(j\omega_0)\beta(j\omega_0)| = 1$. Energy supplied by amplifier exactly replaces energy dissipated per cycle in feedback network.
- Phase Condition: $\angle T(j\omega_0) = \angle A(j\omega_0) + \angle\beta(j\omega_0) = 2\pi n$ $(n = 0, 1, 2, \dots)$. Signals returning through feedback loop arrive perfectly in phase with original input.
- Combined Complex Representation: $A(j\omega_0)\beta(j\omega_0) = 1 + j0 = 1\angle 0^\circ$ at resonant frequency ω_0.

Complex S-Plane Analysis & Pole Trajectories

- Characteristic Equation: System closed-loop transfer function denominator $1 - A(s)\beta(s) = 0$ determines pole locations $s = \sigma \pm j\omega_0$.
- Left-Half Plane ($\sigma < 0$): Closed-loop poles at $s = -\alpha \pm j\omega_0$ produce exponentially damped oscillations $v(t) = V_0 e^{-\alpha t} \cos(\omega_0 t)$ (sub-critical feedback, $|A\beta| < 1$).
- Right-Half Plane ($\sigma > 0$): Closed-loop poles at $s = +\alpha \pm j\omega_0$ produce exponentially growing oscillations $v(t) = V_0 e^{+\alpha t} \cos(\omega_0 t)$ (super-critical feedback, $|A\beta| > 1$).
- $j\omega$-Axis Alignment ($\sigma = 0$): Steady-state Barkhausen condition places poles exactly on imaginary axis at $s = \pm j\omega_0$, maintaining constant amplitude sinusoidal output $v(t) = V_0 \cos(\omega_0 t)$.

Startup Condition vs Steady-State Criterion

- Small-Signal Startup Requirement: At power-up ($t = 0^+$), circuit requires $|A(j\omega_0)\beta(j\omega_0)| > 1$ (typically 1.05 to 3.0) to force poles into RHP and ensure reliable self-starting from thermal noise.
- Exponential Buildup Phase: Output voltage grows according to $v_{out}(t) = V_{noise} \cdot e^{\frac{|A\beta|-1}{2\tau}t} \cdot \sin(\omega_0 t)$.
- Transconductance Saturation: As AC voltage amplitude increases, active device operates across non-linear $I - V$ characteristic, causing large-signal transconductance $G_m(V_{amp})$ to decrease.
- Dynamic Pole Movement: Pole pair shifts leftward from RHP toward $j\omega$-axis, automatically settling at $|A(j\omega_0)\beta(j\omega_0)|_{large-signal} = 1.0$ at target amplitude V_0.

Phase Slope and Frequency Stability Factor

- Frequency Selection via Phase Zero-Crossing: Operating frequency f_0 is fixed where feedback loop phase $\phi(\omega) = \angle A(j\omega) + \angle\beta(j\omega) = 0^\circ$.
- Frequency Stability Factor Definition: $S_F = \omega_0 \frac{d\phi}{d\omega}\Big|_{\omega=\omega_0} = \frac{d\phi}{d(\Delta f/f_0)}\Big|_{f=f_0}$.
- Sensitivity to Phase Noise: A phase perturbation $\Delta\phi$ caused by component variations produces frequency shift $\Delta f_0 \approx \frac{\Delta\phi}{d\phi/d\omega}$.
- High-Q Stabilization: Networks with high phase slope $\frac{d\phi}{d\omega}$ (such as Quartz crystals and high-Q LC tanks) dramatically increase S_F, minimizing frequency drift.

Application Case Study: RC Phase-Shift Oscillator

- Feedback Network Transfer Function: 3-stage high-pass RC ladder transfer factor $\beta(s) = \frac{s^3R^3C^3}{s^3R^3C^3+6s^2R^2C^2+5sRC+1}$.
- Evaluating on $j\omega$ Axis: $\beta(j\omega) = \frac{-j\omega^3R^3C^3}{(1-6\omega^2R^2C^2)+j(5\omega RC-\omega^3R^3C^3)}$.
- Imposing Phase Condition: Phase shift is 180° when imaginary part of denominator equals zero: $5\omega_0RC - \omega_0^3R^3C^3 = 0 \implies \omega_0 = \frac{1}{\sqrt{6}RC}$.
- Imposing Magnitude Condition: Substituting ω_0 into real part gives $\beta(j\omega_0) = -\frac{1}{29}$. To satisfy Barkhausen criterion $A\beta = 1$, inverting amplifier gain must be $A_v = -29$ ($|A_v| = 29$).

Application Case Study: Wien-Bridge Oscillator

- Feedback Network Transfer Function: Lead-lag Wien bridge network $\beta(s) = \frac{Z_p}{Z_s+Z_p} = \frac{sRC}{s^2R^2C^2+3sRC+1}$.
- Evaluating on $j\omega$ Axis: $\beta(j\omega) = \frac{j\omega RC}{(1-\omega^2R^2C^2)+j3\omega RC}$.
- Phase Zero-Crossing Condition: Real term in denominator vanishes at $\omega_0^2R^2C^2 = 1 \implies \omega_0 = \frac{1}{RC} \implies f_0 = \frac{1}{2\pi RC}$.
- Magnitude Criterion Evaluation: At ω_0, $\beta(j\omega_0) = \frac{j\omega_0 RC}{j3\omega_0 RC} = \frac{1}{3}\angle 0^\circ$. Non-inverting op-amp voltage gain $A_v = 1 + \frac{R_f}{R_1}$ must equal 3 ($\frac{R_f}{R_1} = 2$).

Application Case Study: Colpitts LC Oscillator

- Circuit Model: Transistor amplifier coupled with LC tank consisting of split capacitors C_1, C_2 and inductor L.
- Tank Impedance & Resonance: Loop reactance vanishes at $\omega_0 L - \frac{1}{\omega_0 C_1} - \frac{1}{\omega_0 C_2} = 0 \implies \omega_0 = \frac{1}{\sqrt{LC_{eq}}}$, where $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$.
- Feedback Factor Calculation: Voltage division across capacitive divider yields $\beta = \frac{V_f}{V_{out}} = \frac{X_{C1}}{X_{C2}} = \frac{C_2}{C_1}$ (with 180° phase inversion via common-emitter configuration).
- Barkhausen Gain Threshold: Transistor voltage gain requirement $|A_v| \geq \frac{C_1}{C_2}$, or transconductance $g_m R_L' \geq \frac{C_1}{C_2}$.

Non-linear Dynamic Limit Cycles & Transconductance Compression

- Phase-Space Representation: Trajectory in $(v, \frac{dv}{dt})$ state space evolves from origin (noise seed) into a stable closed orbit called a limit cycle.
- BJT Large-Signal Transconductance: Collector current under large AC base drive V_m:
$I_C(t) = I_Q e^{\frac{V_m \cos \omega t}{V_T}} \implies G_m(V_m) = \frac{g_m}{V_m/V_T} \frac{2I_1(V_m/V_T)}{I_0(V_m/V_T)}$.
- Dynamic Self-Balancing: As peak amplitude V_m grows, large-signal transconductance $G_m(V_m)$ continuously decreases until $G_m(V_m)R_L\beta = 1.0$.
- Sinusoidal Purity Tradeoff: Operating near threshold ($A_{start}\beta \approx 1.1$) yields low distortion sinusoidal output; excessive startup gain ($A_{start}\beta \gg 1$) forces severe non-linear clipping.

Limitations of Barkhausen Criterion & Nyquist Encirclement Test

- Linearization Limitation: Barkhausen criterion assumes linear small-signal steady-state conditions; fails to predict non-linear latch-up or multi-frequency squegging.
- Necessity vs Sufficiency: $A\beta = 1\angle 0^\circ$ is a necessary condition for linear systems, but not always sufficient for complex multi-stage circuits with multiple phase crossings.
- Nyquist Stability Test: Plotting polar locus of $A(j\omega)\beta(j\omega)$ from $\omega = -\infty$ to $+\infty$; sustained oscillation requires Nyquist plot to pass through or encircle critical point $(1, j0)$.
- Gain & Phase Margins: Oscillators are engineered with negative gain margin at f_0 during startup to guarantee instability, transitioning to 0 dB margin at steady state.

Phase Noise & Spectral Purity Model

- Spectral Line Broadening: Ideal oscillator produces delta function spectrum $\delta(f - f_0)$; actual oscillator has phase noise sidebands due to thermal and flicker noise.
- Leeson's Phase Noise Model: $L(f_m) = 10\log_{10}\left[\frac{1}{2}\left(1 + \left(\frac{f_0}{2Q_L f_m}\right)^2\right)\left(1 + \frac{f_c}{f_m}\right)\frac{FkT}{P_{sig}}\right]$ in dBc/Hz.
- Key Parameters: f_m is offset frequency from carrier, Q_L is loaded quality factor, f_c is flicker noise corner frequency, F is amplifier noise figure, P_{sig} is signal power.
- Design Rule for High Spectral Purity: Maximize resonator quality factor Q_L and power dissipation P_{sig} while minimizing amplifier noise figure F.

Summary: Key Takeaways on Barkhausen's Criteria

- Barkhausen Conditions: Sustained oscillation requires loop gain magnitude $|A\beta| = 1$ and total loop phase shift $\angle A\beta = 0^\circ$ or 360°.
- Startup vs Steady-State: System requires small-signal $|A\beta| > 1$ for startup from thermal noise, settling to $|A\beta| = 1$ in steady state via gain compression.
- Frequency Determination: Operating frequency f_0 is set by phase condition $\phi(f_0) = 0^\circ$; phase slope $\frac{d\phi}{d\omega}$ dictates frequency stability S_F.
- Practical Applications: Validated across RC Phase-Shift ($f_0 = \frac{1}{2\pi RC\sqrt{6}}, A_v = 29$), Wien-Bridge ($f_0 = \frac{1}{2\pi RC}, A_v = 3$), and Colpitts ($f_0 = \frac{1}{2\pi\sqrt{LC_{eq}}}, A_v = \frac{C_1}{C_2}$).