Basics of Applied Mechanics

Course Code: DI03000201

GTU Slide Creator

Gujarat Technological University

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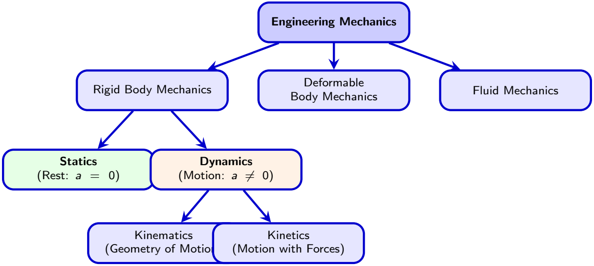
What is Engineering Mechanics?

Engineering Mechanics is the branch of physical science that describes and predicts the state of rest or motion of bodies subjected to the action of forces. It forms the foundational pillar for structural, mechanical, aerospace, and civil engineering design.

- **Engineering Significance:** Allows engineers to design safe structures (bridges, buildings), vehicles (automobiles, aircraft), and machinery without catastrophic failures.
- **Predictive Capability:** Enables accurate computation of internal stresses, deformations, trajectories, and stability under load.
- **GTU Course Context:** DI03000201 – Engineering Mechanics / Applied Mechanics (Unit 1: Basics of Mechanics).

Divisions of Engineering Mechanics

Applied mechanics is subdivided based on the physical state of the body (rigid, deformable, or fluid) and its state of motion (rest or motion).



- **Statics:** Analysis of bodies at rest or moving with constant velocity ($\sum \mathbf{F} = 0, \sum \mathbf{M} = 0$).

- **Dynamics:** Analysis of bodies in accelerated motion ($\sum \mathbf{F} = m\mathbf{a}$).

Basic Entities in Mechanics

Mechanics relies on basic quantities (space, time, mass) and idealized physical models.

- **Space:** Geometric region occupied by bodies, defined by position coordinates (x, y, z).
- **Time:** Measure of succession of events, critical in dynamic analysis.
- **Mass:** Quantity of matter in a body, providing a measure of inertia (resistance to acceleration).
- **Particle:** A body of mass with negligible dimensions; spatial size is neglected, concentrating mass at a single point.
- **Rigid Body:** A combination of a large number of particles where the distance between any two points remains constant under applied forces (zero deformation).

- **Flexible (Deformable) Body:** A body that changes shape under external loads; internal strains and stress distribution must be accounted for.

Mathematical Classification of Physical Quantities

Physical quantities in mechanics are classified based on directional property requirements.

Table: Comparison of Scalar and Vector Quantities in Mechanics

Property	Scalar Quantity	Vector Quantity
Definition	Possesses magnitude only	Possesses magnitude, direction, and obeys vector addition laws
Algebraic Rules	Standard arithmetic rules ($+$, $-$, $\times$)	Vector algebra (Dot product, Cross product, Triangle law)
Engineering Examples	Mass, Time, Volume, Energy, Work, Speed, Density	Force, Velocity, Acceleration, Displacement, Moment, Torque
Representation	Single numerical value with unit (10 kg)	Arrow with magnitude, direction, and point of application

International System of Units (SI)

The standard international metric system defined by seven fundamental base units from which all engineering units are derived.

Table: SI Fundamental (Base) Units in Engineering

Physical Quantity	**Dimension Symbol**	**SI Base Unit**	**Unit Symbol**
Length	$[L]$	Meter	m
Mass	$[M]$	Kilogram	kg
Time	$[T]$	Second	s
Electric Current	$[I]$	Ampere	A
Thermodynamic Temperature	$[\Theta]$	Kelvin	K
Amount of Substance	$[N]$	Mole	mol
Luminous Intensity	$[J]$	Candela	cd

- Primary Base Units in Engineering Mechanics: Meter (m), Kilogram (kg), Second (s).

Formulation of Derived Quantities

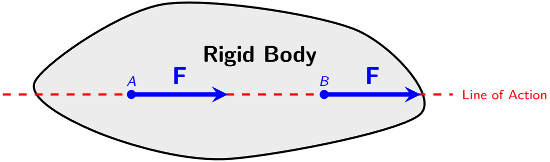
Derived units are expressed as algebraic combinations of fundamental SI base units.

Table: Key Derived Units in Engineering Mechanics

Quantity	Formula Symbol	SI Derived Unit	Base Unit Equivalent	Dimension
Force	$F = m \cdot a$	Newton (N)	$\mathrm{kg \cdot m/s^2}$	$[MLT^{-2}]$
Pressure / Stress	$p = F/A$	Pascal (Pa)	$\mathrm{N/m^2 = kg/(m \cdot s^2)}$	$[ML^{-1}T^{-2}]$
Work / Energy	$W = F \cdot d$	Joule (J)	$\mathrm{N \cdot m = kg \cdot m^2/s^2}$	$[ML^2T^{-2}]$
Power	$P = W/t$	Watt (W)	$\mathrm{J/s = kg \cdot m^2/s^3}$	$[ML^2T^{-3}]$
Moment / Torque	$M = F \cdot r$	Newton-meter ($\mathrm{N \cdot m}$)	$\mathrm{kg \cdot m^2/s^2}$	$[ML^2T^{-2}]$
Density	$\rho = m/V$	$\mathrm{kg/m^3}$	$\mathrm{kg \cdot m^{-3}}$	$[ML^{-3}]$

Principle of Transmissibility of Force

The state of rest or motion of a rigid body remains unchanged if a force acting at a given point is replaced by a force of the same magnitude and direction acting at any other point along its line of action.



- Applies strictly to **Rigid Bodies** for external equilibrium analysis.
- Cannot be applied when analyzing internal stresses or deformable bodies.

Principle of Dimensional Homogeneity

Every term in an engineering equation must have the same physical dimensions. If $[LHS] \neq [RHS]$, the equation is physically invalid.

Table: Common Engineering Unit Conversion Factors

Quantity	**Non-SI Unit**	**SI Equivalent**
Force	1 kgf (Kilogram-force)	9.81 N
Force	1 dyne	10^{-5} N
Pressure	1 bar	10^5 Pa $=$ 100 kPa $=$ 0.1 MPa
Pressure	1 psi (lb/in^2)	6894.76 Pa $\approx$ 6.895 kPa
Work/Energy	1 kWh	3.6×10^6 J $=$ 3.6 MJ

- Always convert non-SI input values into standard SI base units (m, kg, s, N) before carrying out numerical computations.

Summary of Lecture 01

- Mechanics predicts body behavior under forces; divided into Statics ($\mathbf{a} = 0$) and Dynamics ($\mathbf{a} \neq 0$).
- Particle ignores body geometry; Rigid Body neglects deformation; Flexible Body considers deformation.
- Vectors require magnitude, direction, and vector addition compliance.
- SI System relies on 7 base units ($\mathrm{m}, \mathrm{kg}, \mathrm{s}, \ldots$) and derived units ($\mathrm{N}, \mathrm{Pa}, \mathrm{J}, \mathrm{W}$).
- Transmissibility allows moving a force along its line of action for rigid body external equilibrium analysis.
- Dimensional homogeneity ensures physical validity of engineering equations.

Fundamental Definition of Force

Force is defined as an external agency or vector quantity that produces or tends to produce, stops or tends to stop, or changes the velocity/direction of motion of a body, or causes deformation in a deformable body.

- **Newton's Second Law Formulation**: $\vec{F} = m \cdot \vec{a}$, where m is mass (kg) and $\vec{a}$ is acceleration (m/s^2).
- **Dimensional Formula**: $[M^1 L^1 T^{-2}]$ in SI system.
- **Vector Nature**: Possesses both magnitude and direction, obeying parallelogram law of vector addition.

Table: Engineering Units of Force and Conversion Factors

Unit System	Unit Name	Symbol	SI Equivalent / Conversion
SI (Absolute)	Newton	N	$1\,\text{N} = 1\,\text{kg} \cdot \text{m/s}^2$
SI (Multiple)	Kilonewton / Meganewton	kN/MN	$1\,\text{kN} = 10^3\,\text{N}$, $1\,\text{MN} = 10^6\,\text{N}$
Gravitational (Metric)	Kilogram-force	kgf	$1\,\text{kgf} = 9.80665\,\text{N} \approx 9.81\,\text{N}$
Gravitational (CGS)	Gram-force	gf	$1\,\text{gf} = 980.665\,\text{dynes} = 9.81 \times 10^{-3}\,\text{N}$
US Customary	Pound-force	lbf	$1\,\text{lbf} = 4.44822\,\text{N}$

Four Essential Characteristics of a Force

To completely specify a force acting on a rigid body, four distinct parameters must be stated:

1. **Magnitude**: Numerical value representing the size of force with units (e.g., $F = 100\,\mathrm{kN}$).
2. **Direction (Sense & Inclination)**: Line along which the force acts, with arrowhead defining sense (e.g., 45° to horizontal, pointing upwards-right).
3. **Line of Action**: The infinite straight line extending indefinitely along the force vector.
4. **Point of Application**: The exact point on the body where the force is applied (e.g., Point A on beam).

Physical Effects of Force on Bodies

- **Dynamical Effects**: Alters linear velocity (acceleration/deceleration) or causes rotational motion (turning moment $M = F \cdot d$).
- **Statical Effects**: Maintains static equilibrium by balancing opposing external forces.
- **Internal Stress Effects**: Induces tensile, compressive, or shear stresses ($\sigma = F/A$) causing elastic/plastic strains.

Graphical Representation of Force Vector

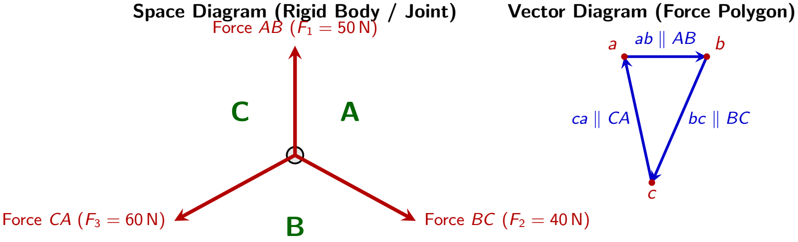
A force is represented graphically by a directed straight line segment ($\vec{AB}$):

- **Length of Line**: Drawn proportional to magnitude according to a chosen scale (e.g., $1\,\text{cm} = 20\,\text{kN}$).
- **Arrowhead**: Indicates the direction and sense of the force.
- **Position**: Indicates the line of action and point of application.

Bow's Notation Method

A standardized graphical system introduced by Robert Henry Bow for identifying forces and drawing force polygons:

- **Space Diagram**: Capital letters (A, B, C, D) are placed in spaces surrounding force lines of action in clockwise sequence.
- **Force Designation**: A force bounded by Space A and Space B is named force AB.
- **Vector Diagram**: Lower-case letters (a, b, c, d) represent vector nodes in the force polygon, where line segment ab represents force AB to scale.



- **Space Diagram**: Bounded regions around lines of action labeled with uppercase letters (A, B, C).
- **Vector Diagram**: Drawn parallel to corresponding space lines of action with lowercase letters (a, b, c). Closed polygon indicates equilibrium ($\sum \vec{F} = 0$).

Statement of the Principle

The state of rest or motion of a rigid body remains unchanged if a force acting at a given point is replaced by another force of the same magnitude and direction acting at any other point along the same line of action.

- **Condition of Applicability**:
  - Body must be perfectly **rigid** (external force effect only).
  - The line of action of the force must remain identical.
- **External Invariance**: Net reaction forces, overall body acceleration, and external rigid-body moments remain completely unchanged when point of application moves along line of action.
- **Critical Exception (Internal Stresses)**:
  - Does **NOT** apply to internal stress distribution or local elastic deformations!

- Example: A pulling force F at the front end of a bar induces tensile stress ($\sigma = +F/A$), whereas pushing with force F at the rear end induces compressive stress ($\sigma = -F/A$).

Statement of Superposition Principle

The combined mechanical effect of a system of forces acting simultaneously on a rigid body is equal to the vector sum of the individual forces acting independently.

- **Equilibrium Addition / Removal Corollary**:
  - The action of a force system on a rigid body is unchanged if another system of forces in equilibrium ($\sum \vec{F} = 0, \sum \vec{M} = 0$) is added to or subtracted from it.
- **Mathematical Formulation**:

$$\vec{R} = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$$

$$R_x = \sum F_{xi}, \quad R_y = \sum F_{yi}, \quad R = \sqrt{R_x^2 + R_y^2}$$

- **Engineering Application**:

- Provides theoretical basis for replacing a complex force system with a single resultant force $\vec{R}$.
- Valid for linear elastic structural systems.

Definition of Force System

When two or more forces act simultaneously on a body, they constitute a **force system**.

- **1. Coplanar Force Systems** (Lines of action lie in a single 2D plane):
  - **Collinear**: Lines of action of all forces lie on the exact same straight line.
  - **Concurrent**: Lines of action of all forces pass through a single common point.
  - **Parallel**: Lines of action are parallel to each other.
    - *Like Parallel*: Forces act in the same direction.
    - *Unlike Parallel*: Forces act in opposite parallel directions.
  - **Non-Concurrent Non-Parallel**: Lines of action do not intersect at a single point and are not parallel.
- **2. Non-Coplanar Force Systems** (Lines of action extend across 3D space):

- **Concurrent**, **Parallel**, or **Non-Concurrent Non-Parallel** in 3D space.

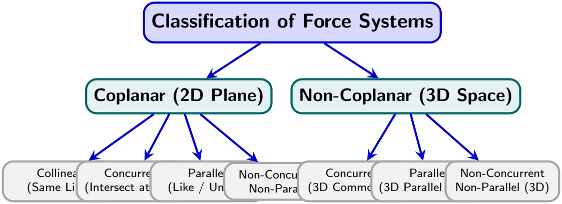

Classification of Force Systems
Coplanar (2D Plane)
Non-Coplanar (3D Space)
Non-Concurrent
Non-Parallel (3D)

Table: Comparison Matrix of Force Systems, Real-World Examples & Equilibrium Conditions

Force System Type	**Geometric Line of Action**	**Real Engineering Example**	**Equilibrium Equations**
Coplanar Collinear	Single common straight line in 2D	Tug-of-war rope, Crane hoist wire rope tension	$\sum F_x = 0$
Coplanar Concurrent	All lines intersect at 1 common point	Crane hook ring, Pin joint in roof truss	$\sum F_x = 0, \sum F_y = 0$
Coplanar Parallel	Lines are parallel in 2D plane	Simply supported bridge girder, Overhead crane beam	$\sum F_y = 0, \sum M = 0$
Coplanar Non-Concurrent	Arbitrary lines in 2D plane	Rigid portal frame, Lathe tool post assembly	$\sum F_x = 0, \sum F_y = 0, \sum M = 0$
Non-Coplanar Concurrent	Intersect at 1 point in 3D space	Camera tripod apex, Guyed antenna tower anchor	$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0$
Non-Coplanar Non-Concurrent	Arbitrary spatial lines in 3D	Aircraft fuselage frame in flight, Space frame truss	$\sum F_{x,y,z} = 0, \sum M_{x,y,z} = 0$

Practical Engineering Applications

- **Bow's Notation**: Fundamental for Maxwell-Cremona graphical statics used in calculating member forces in structural roof trusses and bridge frames.
- **Transmissibility**: Simplifies Free Body Diagram (FBD) construction by allowing relocation of pushing/pulling forces along load lines.
- **Superposition**: Enables structural superposition of beam deflections and multi-load analysis in FEA software.

GTU Examination Key Takeaways

- Define force, list its 4 characteristics (magnitude, direction, line of action, point of application) and SI unit (N).
- Explain Bow's notation convention (A, B, C spaces $\rightarrow a, b, c$ vector nodes).
- Differentiate between Principle of Transmissibility (external rigid body effect invariant) and internal stress distribution.
- State Principle of Superposition and classify force systems with neat sketches/examples.

What is Force Resolution?

Force resolution is the analytical process of decomposing a single force vector into two or more non-interacting components along specified axes, such that their combined vector sum produces the exact physical effect of the original force on a rigid body.

- **Analytical Objective:** Replaces a force acting at an arbitrary angle with equivalent mutually perpendicular (orthogonal) forces.
- **Engineering Utility:** Simplifies complex coplanar force systems, vector addition, and equilibrium calculations ($\Sigma F_x = 0$, $\Sigma F_y = 0$).
- **Course Context:** GTU DI03000201 – Engineering Mechanics (Unit: Coplanar Concurrent Forces).
- **Synthesis vs. Analysis:**

- *Synthesis (Composition):* Combining multiple forces into one resultant force $\vec{R}$.
- *Analysis (Resolution):* Splitting one force $\vec{F}$ into components $\vec{F}_x$ and $\vec{F}_y$.

Rectangular Component Equations

In a two-dimensional Cartesian (x-y) plane, any force $\vec{F}$ inclined at angle θ relative to the horizontal x-axis can be uniquely resolved into rectangular components:

$$F_x = F\cos\theta, \quad F_y = F\sin\theta$$

- **Horizontal Component (F_x):** Projection of $\vec{F}$ along the x-axis (adjacent side).
- **Vertical Component (F_y):** Projection of $\vec{F}$ along the y-axis (opposite side).

- **Magnitude of Original Force:** Reconstructed using Pythagorean theorem:

$$F = \sqrt{F_x^2 + F_y^2}$$

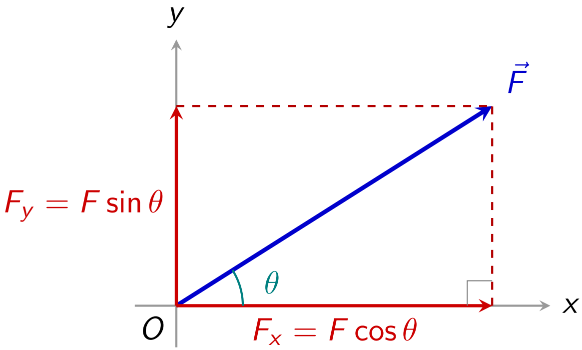
- **Direction Angle (θ):** Calculated via trigonometry:

$$\theta = \tan^{-1}\left(\frac{|F_y|}{|F_x|}\right)$$

- **Vector Cartesian Form:**
  $\vec{F} = F_x\hat{i} + F_y\hat{j} = (F\cos\theta)\hat{i} + (F\sin\theta)\hat{j}$

Geometric View of Orthogonal Decomposition

The figure below illustrates the vector decomposition of force $\vec{F}$ into perpendicular component vectors along the coordinate axes.



- **Right Triangle Property:** The force vector forms the hypotenuse, while F_x and F_y form the leg vectors.
- **Unit Vector Notation:** $\hat{i}$ and $\hat{j}$ represent unit vectors along the positive x and y axes.

Quadrant Dependence of Vector Signs

The algebraic signs of components depend strictly on the quadrant in which the force vector points relative to the origin.

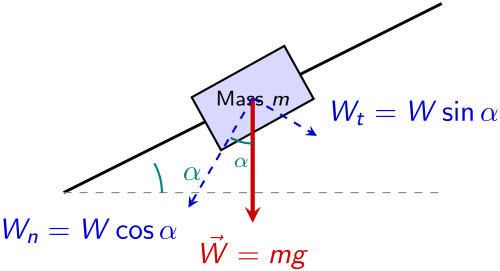
Table: Quadrant Sign Matrix for Orthogonal Components

Quadrant	**Angle Range (θ)**	F_x **Sign**	F_y **Sign**	**Vector Direction**
I	$0^\circ < \theta < 90^\circ$	Positive $(+)$	Positive $(+)$	Right $(\rightarrow)$, Up $(\uparrow)$
II	$90^\circ < \theta < 180^\circ$	Negative $(-)$	Positive $(+)$	Left $(\leftarrow)$, Up $(\uparrow)$
III	$180^\circ < \theta < 270^\circ$	Negative $(-)$	Negative $(-)$	Left $(\leftarrow)$, Down $(\downarrow)$
IV	$270^\circ < \theta < 360^\circ$	Positive $(+)$	Negative $(-)$	Right $(\rightarrow)$, Down $(\downarrow)$

- **Standard Reference Angle:** Angle measured counter-clockwise from the positive x-axis automatically preserves correct signs.
- **Acute Reference Angle:** If using acute angle α with local axis, manually assign signs based on direction arrows.

Normal and Tangential Components

In geotechnical and structural engineering, forces acting on inclined planes (e.g., dams, ramps, trusses) are resolved parallel (t) and perpendicular (n) to the incline angle α.



- **Normal Component (W_n):** $W_n = W \cos\alpha$ (Presses mass into plane).

- **Tangential Component (W_t):** $W_t = W \sin \alpha$ (Tends to slide mass down plane).

Slope Ratio Method ($a : b$)

Engineering drawings frequently define force orientation using a slope triangle with horizontal leg a and vertical leg b rather than an explicit angle.

- **Hypotenuse Calculation:**

$$c = \sqrt{a^2 + b^2}$$

- **Trigonometric Ratios:**

$$\cos\theta = \frac{a}{c} = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin\theta = \frac{b}{c} = \frac{b}{\sqrt{a^2 + b^2}}$$

- **Component Equations:**

$$F_x = F \cdot \left(\frac{a}{c}\right), \quad F_y = F \cdot \left(\frac{b}{c}\right)$$

- **Practical Advantage:** Eliminates rounding errors from inverse trigonometric function evaluations during structural calculations.

Real Structural Engineering Load Analysis

The table below lists actual design forces and calculated orthogonal components across typical structural and mechanical engineering applications.

Table: Orthogonal Components for Structural Engineering Loads

Structural Application	**Force F (kN)**	**Angle θ**	$F_x = F\cos\theta$ **(kN)**	$F_y = F\sin\theta$ **(kN)**
Crane Cable Tension	150.0	30.0°	+129.90	+75.00
Roof Truss Wind Load	45.0	60.0°	+22.50	+38.97
Guy Wire Anchor Load	85.0	135.0°	-60.10	+60.10
Excavator Arm Hydraulic Drag	220.0	210.0°	-190.53	-110.00
Bridge Suspension Cable Pull	310.0	315.0°	+219.20	-219.20
Conveyor Belt Friction Force	12.5	15.0°	+12.07	+3.23

- Note: Negative values indicate components directed along negative coordinate axes (Left for F_x, Down for F_y).

Sample Engineering Problem

A gusset plate at a truss joint is subjected to two cable tension forces:

1. Force $F_1 = 50$ kN directed at $\theta_1 = 35^\circ$ in Quadrant I.
2. Force $F_2 = 80$ kN directed along a $3:4$ slope triangle (3 vertical, 4 horizontal) in Quadrant II.

Calculate the net orthogonal components ΣF_x and ΣF_y acting on the joint.

- **Step 1: Components of F_1 (Quadrant I):**

$$F_{1x} = +50\cos 35^\circ = +40.96 \text{ kN}$$

$$F_{1y} = +50\sin 35^\circ = +28.68 \text{ kN}$$

- **Step 2: Components of F_2 (Quadrant II):** Hypotenuse $c = \sqrt{4^2 + 3^2} = 5$. Since force points in Quadrant II:

$$F_{2x} = -80 \cdot \left(\frac{4}{5}\right) = -64.00 \text{ kN}$$
$$F_{2y} = +80 \cdot \left(\frac{3}{5}\right) = +48.00 \text{ kN}$$

- **Step 3: Summation of Components:**

$$\Sigma F_x = F_{1x} + F_{2x} = 40.96 - 64.00 = -23.04 \text{ kN} \quad (\leftarrow)$$
$$\Sigma F_y = F_{1y} + F_{2y} = 28.68 + 48.00 = +76.68 \text{ kN} \quad (\uparrow)$$

Summary of Key Formulas

- **Orthogonal Decomposition:** $F_x = F\cos\theta$, $F_y = F\sin\theta$.
- **Reconstruction:** $F = \sqrt{F_x^2 + F_y^2}$, $\theta = \tan^{-1}\left(\frac{|F_y|}{|F_x|}\right)$.
- **Inclined Plane:** $W_{\parallel} = W\sin\alpha$, $W_{\perp} = W\cos\alpha$.
- **Slope Triangle ($a : b$):** $F_x = F\left(\frac{a}{c}\right)$, $F_y = F\left(\frac{b}{c}\right)$ where $c = \sqrt{a^2 + b^2}$.

Common Pitfalls to Avoid

- **Axis Reference Mistake:** If angle ϕ is given relative to the vertical y-axis, then $F_x = F\sin\phi$ and $F_y = F\cos\phi$. Always identify the adjacent side!
- **Sign Error:** Always attach correct algebraic signs ($+$ or $-$) based on vector directions along coordinate axes.

Definition of Force Resolution

Resolution of a force is the process of splitting a single given force into two or more components acting along specified directions such that their combined effect on a rigid body is identical to that of the original force.

- **Principle of Transmissibility & Equivalence:** Resolving a force replaces one vector with a system of components having the same net translational and rotational effect.
- **Orthogonal (Rectangular) Components:** When the directions of resolution are mutually perpendicular (typically aligned with Cartesian X and Y axes), the components are termed *orthogonal* or *rectangular* components.

- **Significance in Engineering Mechanics:** Simplifies complex 2D and 3D coplanar concurrent force systems by replacing vector addition with algebraic summation along perpendicular axes.
- **GTU Course Context:** GTU DI03000201 – Engineering Mechanics (Unit 1: Coplanar Concurrent Forces).

Derivation in 2D Cartesian Coordinates

Consider a force $\vec{F}$ of magnitude F acting at an origin O, making an angle θ measured counterclockwise from the positive X-axis.

- **Horizontal Component (F_x):**

$$F_x = F\cos\theta \tag{1}$$

- **Vertical Component (F_y):**

$$F_y = F \sin\theta \tag{2}$$

- **Vector Representation (Unit Vector Notation):**

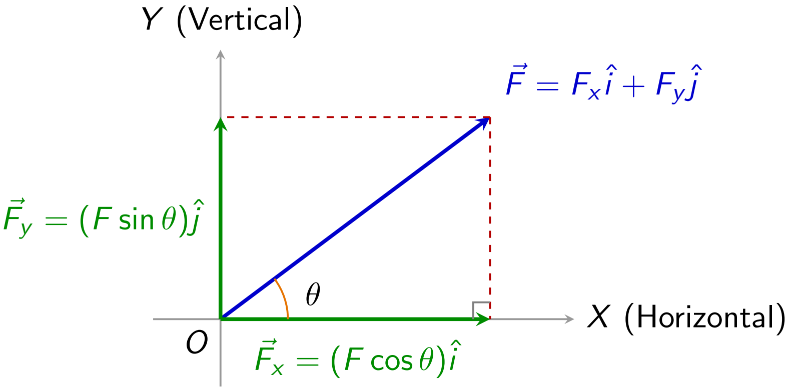
$$\vec{F} = F_x\hat{i} + F_y\hat{j} = (F\cos\theta)\hat{i} + (F\sin\theta)\hat{j} \tag{3}$$

- **Magnitude Reconstitution:**

$$F = |\vec{F}| = \sqrt{F_x^2 + F_y^2} \tag{4}$$

- **Direction Angle (θ):**

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) \tag{5}$$



- Parallelogram / Right-triangle law demonstrates that $\vec{F}$ forms the hypotenuse.
- Adjacent side corresponds to F_x, opposite side corresponds to F_y.

Algebraic Signs Based on Direction

The algebraic sign ($+$ or $-$) of force components depends strictly on the quadrant into which the force vector points.

Table: Quadrant Sign Conventions and Angular Range for Force Components

Quadrant	**Angle Range (θ)**	x-**Component (F_x)**	y-**Component (F_y)**	**Direction ($\vec{F}$)**
I	$0^\circ < \theta < 90^\circ$	Positive ($+$)	Positive ($+$)	Right ($\rightarrow$), Up ($\uparrow$)
II	$90^\circ < \theta < 180^\circ$	Negative ($-$)	Positive ($+$)	Left ($\leftarrow$), Up ($\uparrow$)
III	$180^\circ < \theta < 270^\circ$	Negative ($-$)	Negative ($-$)	Left ($\leftarrow$), Down ($\downarrow$)
IV	$270^\circ < \theta < 360^\circ$	Positive ($+$)	Negative ($-$)	Right ($\rightarrow$), Down ($\downarrow$)

- **Engineering Rule:** Always inspect force directions manually ($\rightarrow +X$, $\leftarrow -X$, $\uparrow +Y$, $\downarrow -Y$) when using acute angles with axes.

Case 1: Angle Measured with Vertical Axis (ϕ)

- If the angle ϕ is specified relative to the Y-axis:

$$F_x = F \sin \phi, \quad F_y = F \cos \phi \tag{6}$$

- *Golden Rule:* The component **adjacent** to the specified angle always takes cos, while the **opposite** component takes sin.

Case 2: Force Direction Specified by Slope Ratio ($a : b$)

- When direction is defined by a slope triangle with horizontal leg a, vertical leg b, and hypotenuse $c = \sqrt{a^2 + b^2}$:

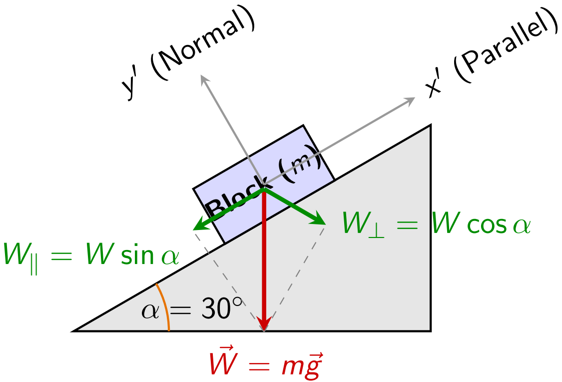
$$F_x = F\left(\frac{a}{c}\right) = F\left(\frac{a}{\sqrt{a^2 + b^2}}\right) \tag{7}$$

$$F_y = F\left(\frac{b}{c}\right) = F\left(\frac{b}{\sqrt{a^2 + b^2}}\right) \tag{8}$$

- Eliminates the need to explicitly calculate the trigonometric angle θ.

Decomposition along Tangential and Normal Axes

In civil and mechanical applications (e.g., vehicles on gradients, block friction, ramp loading), forces are resolved parallel (x') and perpendicular (y') to an inclined surface.



- $W_{\parallel} = W \sin\alpha$ drives sliding down the incline.
- $W_{\perp} = W \cos\alpha$ produces normal reaction force $N = W \cos\alpha$.

Algorithm for Resultant Computation of Coplanar Systems

To determine the single resultant force of several concurrent forces $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$:

1. **Establish Coordinates:** Set up an orthogonal Cartesian reference frame (X, Y) at the point of concurrency.
2. **Identify Reference Angles:** Determine standard counterclockwise angle θ_i or acute angle with X/Y axis for each force.
3. **Calculate Components:** Compute $F_{ix} = F_i \cos\theta_i$ and $F_{iy} = F_i \sin\theta_i$ with proper quadrant sign.

4. **Algebraic Summation:**

$$\Sigma F_x = \sum_{i=1}^{n} F_{ix} = F_{1x} + F_{2x} + \cdots + F_{nx} \tag{9}$$

$$\Sigma F_y = \sum_{i=1}^{n} F_{iy} = F_{1y} + F_{2y} + \cdots + F_{ny} \tag{10}$$

5. **Compute Resultant Magnitude & Angle:**

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}, \quad \theta_R = \tan^{-1}\left|\frac{\Sigma F_y}{\Sigma F_x}\right| \tag{11}$$

Problem Statement

Three forces act at a gusset plate joint: $F_1 = 500$ N at 30°, $F_2 = 750$ N at 135°, and $F_3 = 600$ N at 240°. Calculate the resultant force R and its orientation θ_R.

Table: Resolution and Summation of Concurrent Coplanar System

Force	Magnitude (F)	Orientation (θ)	x-Component ($F_x = F\cos\theta$)	y-Component ($F_y = F\sin\theta$)
$\vec{F}_1$	500 N	30°	$+433.01$ N	$+250.00$ N
$\vec{F}_2$	750 N	135°	-530.33 N	$+530.33$ N
$\vec{F}_3$	600 N	240°	-300.00 N	-519.62 N
Total	—	—	$\Sigma F_x = -397.32$ **N**	$\Sigma F_y = +260.71$ **N**

- **Resultant Magnitude:**

$$R = \sqrt{(-397.32)^2 + (260.71)^2} = \sqrt{157863.18 + 67969.70} = 475.22 \text{ N} \tag{12}$$

- **Resultant Direction:**

$$\theta' = \tan^{-1}\left(\frac{260.71}{397.32}\right) = 33.27^\circ \quad \text{(in Quadrant II, } \leftarrow\uparrow \text{)} \tag{13}$$

$$\theta_R = 180^\circ - 33.27^\circ = 146.73^\circ \quad \text{(from } +X \text{ axis)} \tag{14}$$

Key Real-World Applications:

- **Truss & Bridge Joint Analysis:** Resolving member forces at nodes to solve equilibrium equations.
- **Cranes & Cable Suspension:** Computing tension components in hoisting cables and boom supports.
- **Robotic Link Mechanics:** Decomposing actuator forces into radial and tangential joint efforts.

Common Student Errors to Avoid:

1. **Blindly applying $\cos$ to X:** Always check which axis forms the given reference angle!
2. **Ignoring Vector Signs:** Omitting negative signs for leftward ($\leftarrow$) or downward ($\downarrow$) forces during summation.
3. **Calculator Mode Errors:** Mixing up Radians and Degrees during trigonometric calculations.

4. **Incorrect Quadrant Placement:** Failing to adjust $\tan^{-1}(|F_y/F_x|)$ to the true global angle based on component signs.

Definition of Equilibrium

A body or system of coplanar concurrent forces is said to be in **static equilibrium** if the vector sum of all forces acting on it is equal to zero, causing no change in its state of rest or uniform motion.

- **Resultant Force Condition:** Static equilibrium requires that the resultant force $\vec{R} = \mathbf{0}$.
- **Concurrent Force Characteristics:** Since all force vectors intersect at a single point, no rotational moment ($\sum M = 0$) is generated around the concurrency node.
- **Engineering Significance:** Structural node joints, cable-stayed bridges, crane hooks, and truss gusset plates rely on static equilibrium analysis to prevent structural failure.
- **Newton's First Law Connection:** An object at rest remains at rest when net external coplanar force is zero ($\sum \vec{F} = \mathbf{0}$).

What is an Equilibrant Force?

The **equilibrant force** ($\vec{E}$) is defined as a single force which, when added to a system of coplanar forces, brings the entire force system into static equilibrium.

- **Magnitude:** The equilibrant has a magnitude equal to the resultant force ($|\vec{E}| = |\vec{R}|$).
- **Direction:** The equilibrant acts in the exact opposite direction to the resultant ($\theta_E = \theta_R + 180^\circ$).
- **Line of Action:** It acts along the same line of action passing through the concurrency point.

Mathematical Comparison

$$\text{Resultant Vector:} \quad \vec{R} = \left(\sum F_x\right)\hat{i} + \left(\sum F_y\right)\hat{j}$$

$$\text{Equilibrant Vector:} \quad \vec{E} = -\vec{R} = -\left(\sum F_x\right)\hat{i} - \left(\sum F_y\right)\hat{j}$$

Equations of Equilibrium for Coplanar Concurrent System

For a system of forces lying in the *xy*-plane and intersecting at a single point, the necessary and sufficient conditions for equilibrium are:

1. **Algebraic Sum of Horizontal Components:**
$$\sum F_x = F_{1x} + F_{2x} + \cdots + F_{nx} = 0 \quad (\text{or } \sum F\cos\theta = 0)$$
2. **Algebraic Sum of Vertical Components:**
$$\sum F_y = F_{1y} + F_{2y} + \cdots + F_{ny} = 0 \quad (\text{or } \sum F\sin\theta = 0)$$

- **Magnitude of Resultant:**
$$R = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2} = \sqrt{0^2 + 0^2} = 0$$
- **Graphical Criterion:** The force polygon formed by consecutive vector addition of all active forces must form a closed polygon.

Definition of Free-Body Diagram

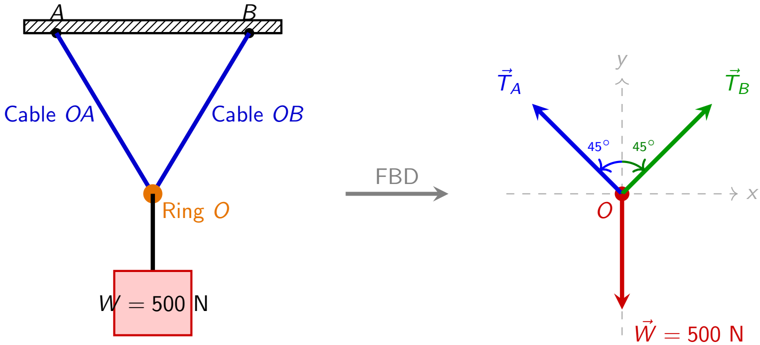
A **Free-Body Diagram (FBD)** is a graphical representation of an isolated body or node, completely separated from its mechanical surroundings, displaying all active external forces, reactive contact forces, and gravity forces.

- **Steps to Construct an FBD:**
  1. **Isolate:** Select the specific node or body under equilibrium and draw its outline.
  2. **Active Forces:** Draw all known applied loads, weights ($W = mg$), and external pulls.
  3. **Reactive Forces:** Replace all physical supports, cables, and surfaces with reaction forces (Cable Tension T, Normal Force N, Friction f).
  4. **Coordinate System & Angles:** Establish x-y axes and label all geometric angles.

- **Crucial Rule:** Cables always exert a *pulling force* (tension directed away from the body/node).

Visualizing a Suspended Cable Ring Connection

Consider a heavy load $W = 500$ N suspended from a central ring O held by cables OA and OB.



(a) Physical Setup **(b) Free-Body Diagram**

Statement of Lami's Theorem

If three coplanar concurrent forces acting at a point are in static equilibrium, each force is directly proportional to the sine of the angle between the remaining two forces.

Mathematical Formulation

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

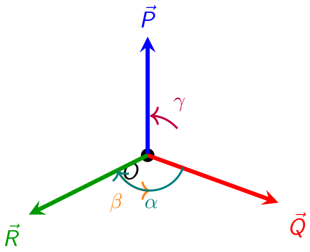
where:

- P, Q, R are the magnitudes of the three concurrent forces.
- α is the angle between forces Q and R.
- β is the angle between forces P and R.
- γ is the angle between forces P and Q.

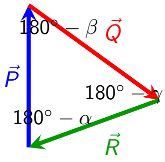
- **Prerequisites for Lami's Theorem:**
  1. Exactly three forces (no more, no less).
  2. All forces must be concurrent and coplanar.
  3. All forces must be directed either outward (away from point) or inward (towards point).

Geometric Proof and Closed Vector Polygon

Lami's Theorem is derived directly from the Sine Rule applied to a closed force triangle in static equilibrium ($\vec{P} + \vec{Q} + \vec{R} = \mathbf{0}$).



(a) Concurrent Force System



(b) Closed Triangle ($\sum \vec{F} = \mathbf{0}$)

Real Engineering Test Data (Universal Force Board Experiment)

Experimental validation of coplanar concurrent equilibrium using three string tension hangers and a center ring.

Exp. No.	**Force F_1 (N)**	**Angle θ_1 (deg)**	**Force F_2 (N)**	**Angle θ_2 (deg)**	**Theoretical E_{th} (N)**	**Measured E_{exp} (N)**
1	10.0	0°	15.0	60°	21.79	21.80
2	25.0	30°	35.0	120°	43.01	42.90
3	40.0	45°	60.0	135°	72.11	72.50
4	50.0	0°	50.0	90°	70.71	70.60
5	75.0	15°	100.0	105°	140.09	139.80

Table: Experimental vs Analytical Equilibrant Values (Percentage Error $< 0.6\%$).

- **Observation:** Experimental equilibrant force magnitude E_{exp} closely matches analytical value $E_{th} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos(\theta_2 - \theta_1)}$.
- **Sources of Error:** String friction over pulleys, scale resolution limits (± 0.1 N), and ring mass weight.

Standard Problem-Solving Methodology

To solve any coplanar concurrent equilibrium problem in engineering mechanics, follow these 6 structured steps:

1. **Identify Node of Concurrency:** Locate the single point where lines of action of all unknown and known forces intersect.
2. **Draw Free-Body Diagram (FBD):** Isolate the node and draw all active forces, tension vectors, and normal reaction vectors.
3. **Set Cartesian Axes:** Orient reference x- and y-axes logically (usually horizontal and vertical).
4. **Resolve Forces into Components:** Calculate x-components ($F \cos \theta$) and y-components ($F \sin \theta$) paying strict attention to signs ($+x, -x, +y, -y$).
5. **Apply Equilibrium Equations:**

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0$$

(Alternatively, apply Lami's theorem if exactly 3 non-collinear forces act at the node).

6. **Solve Simultaneous Equations:** Compute unknown force magnitudes or geometric angles, verifying physical reality (e.g., tension $T > 0$).

Coplanar Concurrent Force System

A system of forces is termed **coplanar concurrent** when the lines of action of all forces lie in the same two-dimensional plane and intersect at a single common point (point of concurrence).

- **Definition of Resultant Force (R):** A single equivalent force that produces the exact same kinematic and dynamic effect on a rigid body as the entire system of original forces acting together.
- **Methods of Finding Resultant for Point Forces:**
  - **Analytical Methods:** Mathematical calculation using trigonometric principles and vector component resolution ($\sum F_x, \sum F_y$).
  - **Graphical Methods:** Geometric vector diagrams drawn to a chosen physical scale (e.g., 1 cm $=$ 100 N).
- **Fundamental Laws of Force Addition:**

1. Law of Parallelogram of Forces (2 forces)
2. Law of Triangle of Forces (2 forces)
3. Law of Polygon of Forces (3 or more forces)

Statement of Parallelogram Law

If two coplanar forces acting simultaneously at a single point are represented in magnitude and direction by two adjacent sides of a parallelogram drawn from that point, their **resultant** is represented in magnitude and direction by the **diagonal** of the parallelogram passing through the same point.

- Let two forces F_1 and F_2 act at point O with an included angle θ.
- **Analytical Formula for Magnitude (R):**

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

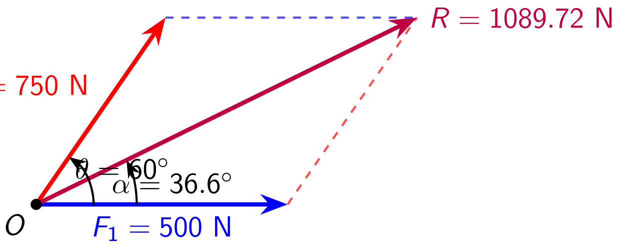
- **Analytical Formula for Direction (α relative to F_1):**

$$\tan\alpha = \frac{F_2\sin\theta}{F_1 + F_2\cos\theta} \implies \alpha = \tan^{-1}\left(\frac{F_2\sin\theta}{F_1 + F_2\cos\theta}\right)$$

- **Special Angle Conditions:**
  - When $\theta = 0^\circ$ (Collinear, same direction): $R = F_1 + F_2$
  - When $\theta = 90^\circ$ (Perpendicular): $R = \sqrt{F_1^2 + F_2^2}$
  - When $\theta = 180^\circ$ (Collinear, opposite direction): $R = |F_1 - F_2|$

Engineering Problem Example

A crane joint is subjected to two concurrent structural cable forces: $F_1 = 500$ N horizontally and $F_2 = 750$ N at an angle $\theta = 60^\circ$ to F_1. Determine the resultant magnitude R and inclination angle α.



- **Calculation Steps:**

$$R = \sqrt{500^2 + 750^2 + 2(500)(750)\cos 60^\circ} = \sqrt{250000 + 562500 + 375000} = \sqrt{1187500} = 1089.72 \text{ N}$$

$$\tan\alpha = \frac{750\sin 60^\circ}{500 + 750\cos 60^\circ} = \frac{649.52}{875} = 0.7423 \implies \alpha = 36.59^\circ$$

Experimental Verification & Error Analysis

In engineering laboratory verification, force table measurements are compared against analytical calculations for various force combinations.

Table: Engineering Data: Analytical vs Graphical Resultants (1 cm = 100 N Scale)

Case	F_1 **(N)**	F_2 **(N)**	θ **(°)**	**Analytical** R **(N)**	**Graphical** R **(N)**	**Error (%)**
1	300	400	90	500.00	502.0	0.40%
2	500	750	60	1089.72	1092.5	0.25%
3	600	600	120	600.00	598.0	0.33%
4	450	800	45	1161.42	1158.0	0.29%
5	700	350	135	508.82	511.0	0.43%

- **Precision Drivers:** Graphical error is minimized by using sharp pencil leads, large drawing scales, and high-precision drafting protractors.
- **Engineering Utility:** Graphical methods provide rapid visual sanity checks during structural design.

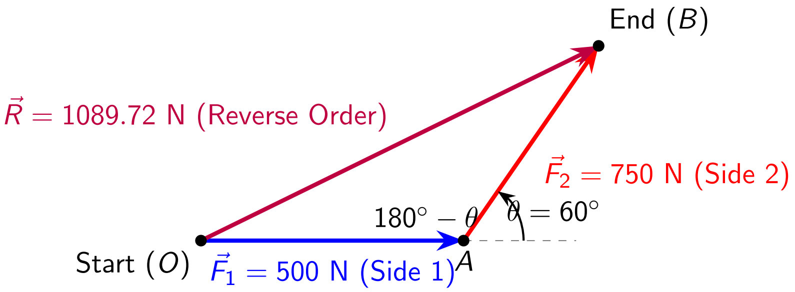
Statement of Triangle Law

If two coplanar forces acting simultaneously at a point are represented in magnitude and direction by two sides of a triangle taken in **head-to-tail order**, their resultant is represented in magnitude and direction by the **third side** of the triangle taken in **reverse order** (from tail of first vector to head of second vector).

- **Geometric Vector Addition:** $\vec{R} = \vec{F}_1 + \vec{F}_2$
- **Trigonometric Sine Rule (Lami's Theorem precursor):**
$$\frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha} = \frac{R}{\sin(180^\circ - \theta)}$$
- **Trigonometric Cosine Rule on Vector Triangle:**
$$R^2 = F_1^2 + F_2^2 - 2F_1F_2\cos(180^\circ - \theta) = F_1^2 + F_2^2 + 2F_1F_2\cos\theta$$
- Triangle law forms the foundation for graphic statics and truss vector diagrams.

Head-to-Tail Construction

The force vectors are placed end-to-end preserving magnitude and angle. The closing side from origin to endpoint forms the resultant vector $\vec{R}$.



- Notice that vector $\vec{F}_1$ goes $O \to A$ and vector $\vec{F}_2$ goes $A \to B$ in order.
- The resultant vector $\vec{R}$ goes $O \to B$ in reverse order (closing the vector triangle).

Statement of Polygon Law

If any number of coplanar concurrent forces acting at a point are represented in magnitude and direction by the sides of an open polygon taken in order, their **resultant** is represented in magnitude and direction by the **closing side** of the polygon taken in **reverse order**.

- **Graphical Construction Procedure:**
  1. Select a suitable linear force scale (e.g., 1 cm $=$ 50 N).
  2. Draw vector $\vec{F}_1$ from initial origin O parallel to its direction.
  3. From the head of $\vec{F}_1$, draw vector $\vec{F}_2$ parallel to its line of action.
  4. Continue appending vectors $\vec{F}_3, \vec{F}_4, \ldots, \vec{F}_n$ in head-to-tail order.
  5. Join the initial starting point O to the tip of the final vector $\vec{F}_n$.
  6. Measure length of closing side (magnitude $=$ length $\times$ scale) and measure angle with reference axis.

Multi-Force Crane Hook Node Problem

Four coplanar concurrent forces act on a gusset plate node at origin O: $F_1 = 200$ N at 0°, $F_2 = 300$ N at 45°, $F_3 = 400$ N at 120°, $F_4 = 150$ N at 240°.

Table: Component Resolution of 4-Force Coplanar Concurrent System

Force	**Mag. (N)**	**Angle** θ_i	$F_x = F_i \cos\theta_i$ **(N)**	$F_y = F_i \sin\theta_i$ **(N)**
F_1	200	0°	$+200.00$	0.00
F_2	300	45°	$+212.13$	$+212.13$
F_3	400	120°	-200.00	$+346.41$
F_4	150	240°	-75.00	-129.90
Total	–	–	$\sum F_x = +137.13$	$\sum F_y = +428.64$

- **Analytical Resultant Calculation:**

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2} = \sqrt{(137.13)^2 + (428.64)^2} = \sqrt{18804.6 + 183732.2} = 450.04 \text{ N}$$

$$\theta_R = \tan^{-1}\left(\frac{\sum F_y}{\sum F_x}\right) = \tan^{-1}\left(\frac{428.64}{137.13}\right) = 72.26^\circ \text{ (Quadrant I)}$$

Method Selection Guide for Coplanar Concurrent Forces

Choosing the optimal solution technique depends on number of forces and required numerical precision.

Table: Comparison of Force Resultant Methods

Criterion	Parallelogram Law	Triangle Law	Polygon Law
Number of Forces	Exactly 2 forces	Exactly 2 forces	3 or more forces
Primary Method	Analytical / Graphical	Graphical / Analytical	Graphical / Vector addition
Key Formula / Tool	$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$	Sine/Cosine Rule	$\sum F_x, \sum F_y$ Resolution
Engineering Use	Cable tensions, 2-bar joints	Vector space diagrams	Structural gusset plates, crane hooks

- **GTU Exam Key Takeaways:**
    - Always state the scale clearly when drawing graphical force polygons.
    - Remember: Resultant vector closes the triangle / polygon in **reverse order**.
    - Check component signs ($\pm F_x$, $\pm F_y$) carefully based on quadrant angles.

- **Course Context:** GTU Course DI03000201 – Engineering Mechanics.
- **Unit Focus:** Coplanar Concurrent Force Systems.
- **Key Definitions:**
  - **Coplanar Forces:** Forces whose lines of action all lie in the same two-dimensional plane.
  - **Concurrent Forces:** Forces whose lines of action pass through a single common point of intersection.
  - **Resultant Force (R):** A single hypothetical force that produces the exact same external dynamic or static effect on a rigid body as the entire system of individual forces combined.
- **Methods of Determination:**
  - **Analytical Methods:** Mathematical calculation using vector trigonometry and component resolution (Law of Parallelogram, Sine/Cosine Laws, Orthogonal Resolution).
  - **Graphical Methods:** Geometric vector diagrams drawn to scale (Law of Triangle, Law of Polygon).

- **Statement of the Law:**
  - If two concurrent forces acting simultaneously at a point are represented in magnitude and direction by two adjacent sides of a parallelogram drawn from that point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through the same point.
- **Analytical Formulae:**
  - **Magnitude of Resultant (R):**

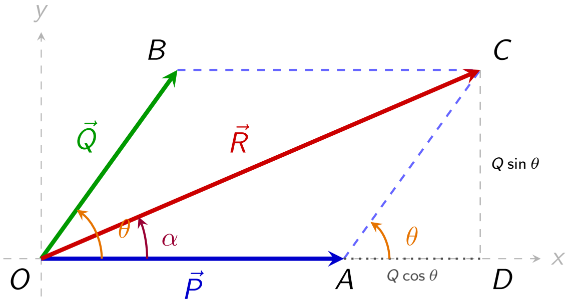
    $$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

    where P and Q are force magnitudes, and θ is the included angle between their lines of action.
  - **Direction of Resultant (α with respect to force P):**

    $$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta} \implies \alpha = \arctan\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right)$$
- **Special Engineering Cases:**
  - Collinear Same Direction ($\theta = 0^\circ$): $R = P + Q$, $\alpha = 0^\circ$.

- Perpendicular Forces ($\theta = 90°$): $R = \sqrt{P^2 + Q^2}$, $\tan \alpha = Q/P$.
- Collinear Opposite Direction ($\theta = 180°$): $R = |P - Q|$.
- Equal Forces ($P = Q$): $R = 2P \cos(\theta/2)$, $\alpha = \theta/2$.

- Geometric construction for concurrent forces $\vec{P}$ and $\vec{Q}$ acting at origin O with angle θ:



- From right-angled triangle $\triangle OCD$:

$$OC^2 = OD^2 + CD^2 = (OA + AD)^2 + CD^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$R^2 = P^2 + 2PQ\cos\theta + Q^2(\cos^2\theta + \sin^2\theta) = P^2 + Q^2 + 2PQ\cos\theta$$

- Real engineering application: Design analysis of a heavy lifting crane eye-bolt loaded by two cable tensions $P = 15$ kN and $Q = 25$ kN.
- Evaluating resultant force R and equivalent normal stress σ across shank area $A = 500$ mm^2 for different inter-cable angles θ:

Table: Parallelogram Law Calculation for Crane Eye-Bolt Joint ($P = 15$ kN, $Q = 25$ kN)

Angle θ ($^\circ$)	**Resultant R (kN)**	**Direction α ($^\circ$)**	**Stress σ (MPa)**	**Structural Assessment**
30°	38.73	18.81°	77.46	Peak tensile load; check bolt yield strength.
60°	35.00	38.21°	70.00	Moderate tensile load; standard operation zone.
90°	29.15	59.04°	58.31	Orthogonal cables; lower resultant magnitude.
120°	21.79	83.41°	43.59	High force cancellation; reduced bolt stress.
150°	13.79	114.93°	27.58	Opposing tension configuration; minimum net load.

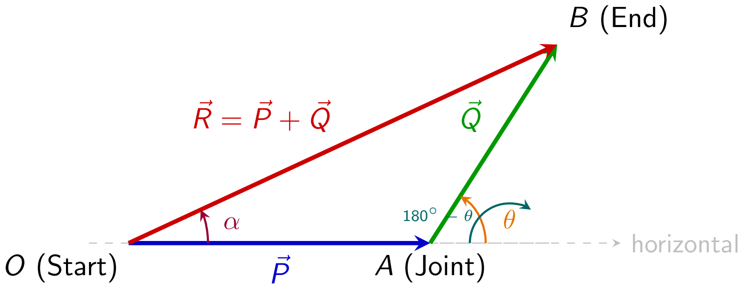
- **Statement of Triangle Law:**
  - If two concurrent forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in order (tip-to-tail), their resultant is represented in magnitude and direction by the third side (closing side) taken from the starting point to the final head.
- **Equilibrium Form of Triangle Law:**
  - If three concurrent forces acting at a point are in static equilibrium, they can be represented in magnitude and direction by the three sides of a closed triangle taken in order.
- **Analytical Trigonometric Relations (Law of Sines):**

$$\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin(180^\circ - \theta)}$$

- **Analytical Law of Cosines on Force Triangle:**

$$R^2 = P^2 + Q^2 - 2PQ\cos(180^\circ - \theta) = P^2 + Q^2 + 2PQ\cos\theta$$

- Tip-to-Tail graphical construction of resultant vector $\vec{R} = \vec{P} + \vec{Q}$:



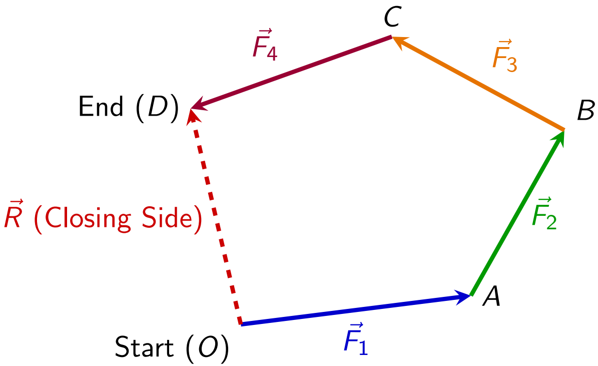
- **Graphical Construction Steps:**
  1. Select scale factor 1 cm $= k$ N.
  2. Draw vector $\vec{OA} = \vec{P}$ parallel to line of action.
  3. From head A, draw vector $\vec{AB} = \vec{Q}$ at angle θ.
  4. Measure closing vector $\vec{OB} = \vec{R}$ and angle α.

- **Statement of Polygon Law:**
  - If any number of coplanar concurrent forces acting at a point are represented in magnitude and direction by the sides of an open polygon taken in order, their resultant is represented in magnitude and direction by the closing side of the polygon drawn from the starting point to the final head.
- **Condition of Static Equilibrium:**
  - If the force polygon closes completely (the head of the last force vector coincides with the tail of the first force vector), the resultant force $R = 0$, meaning the concurrent force system is in static equilibrium:

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0$$

- **Generalization:**
  - Law of Polygon is the extension of the Law of Triangle for systems with 3 or more concurrent forces ($N \geq 3$).

- Graphical vector addition of 4 concurrent forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \vec{F}_4$ forming an open force polygon:



- Vector relationship: $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$.

- **Orthogonal Resolution Procedure for N Concurrent Forces:**
  - Resolve each force F_i into horizontal (X) and vertical (Y) components using inclination angle θ_i with positive X-axis:
    $$F_{ix} = F_i \cos\theta_i, \quad F_{iy} = F_i \sin\theta_i$$
  - Compute algebraic sum of components along X and Y axes:
    $$\sum F_x = \sum_{i=1}^{N} F_i \cos\theta_i, \quad \sum F_y = \sum_{i=1}^{N} F_i \sin\theta_i$$
  - Calculate resultant magnitude R and orientation angle θ_R:
    $$R = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2}, \quad \theta_R = \arctan\left|\frac{\sum F_y}{\sum F_x}\right|$$
- **Verification Case Study (4-Force Truss Joint System):**

Table: Resolution vs Graphical Polygon for 4-Force Joint ($F_1 = 120$ N at 0°, $F_2 = 200$ N at 45°, $F_3 = 150$ N at 120°, $F_4 = 100$ N at 240°)

Parameter	**Analytical Method**	**Graphical Polygon**	**Percentage Error**
$\sum F_x$ (Horizontal Sum)	$+136.42$ N	$+136.00$ N	0.31%
$\sum F_y$ (Vertical Sum)	$+184.72$ N	$+185.00$ N	0.15%
Resultant Magnitude (R)	229.64 N	230.00 N	0.16%
Resultant Angle (θ_R)	53.55°	53.50°	0.09%

- **Comparison Matrix of Methods for Concurrent Forces:**

Table: Comparative Analysis of Analytical and Graphical Force Methods

Method	**Applicability**	**Primary Advantage**	**Limitations**
Law of Parallelogram	Exactly 2 concurrent forces	Direct analytical formula for R and α	Cumbersome for $>$ 2 forces.
Law of Triangle	Exactly 2 forces (or 3 in equilibrium)	Visual vector addition; direct Sine Law application	Requires accurate drafting scale for graphical solution.
Law of Polygon	Any number of concurrent forces ($N \geq 3$)	Excellent for quick graphical equilibrium check	Cumulative drafting errors affect final closing side.
Method of Components	Arbitrary number of concurrent forces	High precision numerical output for CAD/FEA software	Requires trigonometric sign conversion by quadrant.

- **Engineering Recommendation:** Use analytical component resolution for design calculations, and validate with graphical polygon sketch during field inspection.

- **Course Context:** GTU Course DI03000201 – Engineering Mechanics / Applied Mechanics.
- **Unit II:** Coplanar Concurrent Force Systems.
- **Coplanar Concurrent Forces:**
  - A system of forces lying in a single two-dimensional plane (xy-plane) whose lines of action pass through a common single point (point of concurrence O).
  - Practical examples: Cables connected at a central ring, guy wires on a radio tower, smooth spheres resting on channel boundaries.
- **Conditions of Static Equilibrium:**
  - For a particle or rigid body at rest under concurrent forces, the net resultant force $\vec{R}$ must be zero:

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \quad \implies \quad \vec{R} = 0 \tag{15}$$

- When exactly three coplanar concurrent forces act on a body in equilibrium, Lami's Theorem offers a direct trigonometric method without component resolution.

- **Statement of Lami's Theorem:**
  - If three coplanar concurrent forces acting at a point keep a body in static equilibrium, then each force magnitude is directly proportional to the sine of the angle between the other two forces.
- **Mathematical Formulation:**

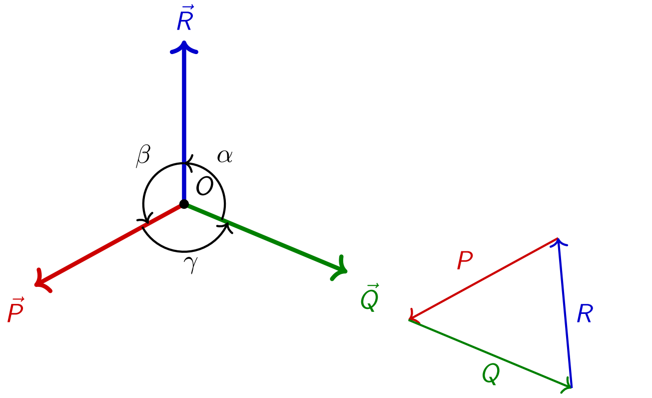
$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \tag{16}$$

where:

- P, Q, R are the magnitudes of the three concurrent forces.
- α is the angle between forces Q and R (opposite to force P).
- β is the angle between forces P and R (opposite to force Q).

- γ is the angle between forces P and Q (opposite to force R).
- Note: $\alpha + \beta + \gamma = 360°$.

- **Essential Conditions for Validity:**
  - Exactly **three** forces must act at the concurrent point.
  - Forces must be **coplanar** and **concurrent**.
  - Forces must be directed either all **away from** the node (tensile) or all **towards** the node (compressive).



Triangle of Forces (Closed)

- **Proof via Sine Rule (Triangle of Forces):**

- Since the three concurrent forces keep the particle in equilibrium, their vector sum forms a closed force triangle.
- Applying the Law of Sines to the closed triangle:

$$\frac{P}{\sin(180^\circ - \alpha)} = \frac{Q}{\sin(180^\circ - \beta)} = \frac{R}{\sin(180^\circ - \gamma)} \tag{17}$$

- Using trigonometric identity $\sin(180^\circ - \theta) = \sin\theta$, we obtain Lami's equation:

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \tag{18}$$

- **Step 1: Identify the Concurrent Node**
  - Locate the specific point where the lines of action of all three forces intersect (e.g., cable knot, sphere center, hinge pin).
- **Step 2: Draw the Free Body Diagram (FBD)**
  - Isolate the node and draw all three force vectors with accurate line directions.
  - Ensure all vectors point away from the node (pull) or all towards it (push).
- **Step 3: Determine Internal Angles**
  - Calculate angles α, β, γ between adjacent force vectors using geometry and trigonometry.
  - Verify that internal angle sum equals 360° ($\alpha + \beta + \gamma = 360°$).
- **Step 4: Substitute into Lami's Ratio**
  - Set up the proportion: $\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$.
  - Equate pairs of ratios to compute unknown force magnitudes or unknown angles.

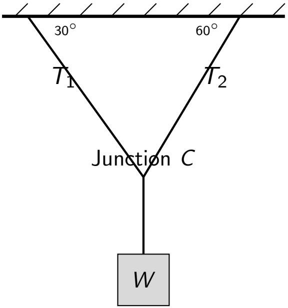
- **1. Cable and Rigging Systems:**
  - Tensions in overhead crane hoist cables, guy wires on telecommunication towers, and suspended lamp posts.
- **2. Spheres and Cylinders on Inclined Planes:**
  - Surface reactions on pressure vessels, boilers, storage cylinders, and ball bearings resting in V-grooves or channels.
- **3. Pin-Jointed Structural Nodes:**
  - Internal axial tension and compression forces in 3-bar wall brackets and tripod supports.
- **4. Machine Tool Clamping:**
  - Contact force distribution in self-centering V-blocks and mechanical wedge clamps.

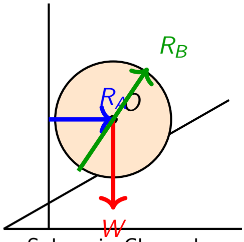
- **Problem Statement:**
    - A payload mass $m = 50$ kg (weight $W = 50 \times 9.81 = 490.5$ N) is suspended at junction C by two support cables AC and BC.
    - Cable AC makes 30° with the horizontal ceiling, and cable BC makes 60° with the horizontal ceiling.
    - Determine the tension forces T_1 (in cable AC) and T_2 (in cable BC).
- **Angle Calculations at Junction C:**
    - Angle opposite T_2 (α between T_1 and W): $\alpha = 90^\circ + 30^\circ = 120^\circ$.
    - Angle opposite T_1 (β between T_2 and W): $\beta = 90^\circ + 60^\circ = 150^\circ$.
    - Angle opposite W (γ between T_1 and T_2): $\gamma = 180^\circ - (30^\circ + 60^\circ) = 90^\circ$.

- **Lami's Theorem Application:**

$$\frac{T_1}{\sin 120^\circ} = \frac{T_2}{\sin 150^\circ} = \frac{490.5}{\sin 90^\circ} \tag{19}$$

- **Final Computation:**
  - $T_1 = 490.5 \times \sin 120^\circ = 490.5 \times 0.8660 = 424.79$ N.
  - $T_2 = 490.5 \times \sin 150^\circ = 490.5 \times 0.5000 = 245.25$ N.





Sphere in Channel

- **Problem Case: Industrial Piping Sling Analysis**
  - Engineering analysis of cable tension distribution for a heavy chemical pipe suspended by two sling cables under static equilibrium.
  - Computed tension magnitudes T_1, T_2 for varying sling angles θ_1, θ_2 and payloads W:

Case	Payload W (kN)	Angle θ_1	Angle θ_2	Tension T_1 (kN)	Tension T_2 (kN)
1	10.0	45°	45°	7.07	7.07
2	10.0	30°	60°	8.66	5.00
3	15.0	30°	30°	15.00	15.00
4	20.0	60°	60°	11.55	11.55
5	25.0	15°	75°	24.15	6.47

Table: Parametric Cable Tension Data Calculated via Lami's Theorem

- **Engineering Insights:**
  - Decreasing sling angle θ below 30° sharply increases tension T_1 due to $\frac{1}{\sin\theta}$ amplification.

- Symmetric sling arrangements ($\theta_1 = \theta_2$) equalize load distribution ($T_1 = T_2$).

Feature / Parameter	Lami's Theorem	Method of Resolution ($\sum F_x, \sum F_y$)
Applicability	Strictly 3 coplanar concurrent forces	Any number of coplanar forces ($N \geq 2$)
Computational Steps	Single step proportional equation	Resolving into x, y components & 2 equations
Trigonometric Function	Uses Sine Rule ($\sin\theta$)	Uses $\cos\theta$ and $\sin\theta$ components
Direction Requirement	All forces must point away/towards	Components handle direction signs automatically
Calculation Speed	Very fast for 3-force systems	Moderate; required for > 3 force systems
Error Susceptibility	Low (direct single calculation)	Moderate (sign convention errors in components)

Table: Comparative Assessment of Equilibrium Analysis Methods

- **Summary Recommendation for GTU Examinations:**
  - Apply Lami's Theorem whenever a problem involves exactly three concurrent forces in equilibrium.
  - Utilize the rectangular component method ($\sum F_x = 0, \sum F_y = 0$) for force systems with four or more forces.

Fundamental Definitions

- **Center of Gravity (CG):** The point through which the resultant gravitational force (total weight W) of a body acts, regardless of the body's orientation.
- **Centroid (C):** The geometric center of a 2D plane area, 1D curve, or 3D volume; the point where the entire geometric measure is imagined to be concentrated.
- **Center of Mass (CM):** The point where the total mass of a body is concentrated for analyzing translational dynamics.

- **Physical vs. Geometric Concepts:**
  - CG applies to physical bodies with mass subject to a gravitational field ($W = m \cdot g$).

- Centroid is purely a geometric property depending solely on shape and dimensions.

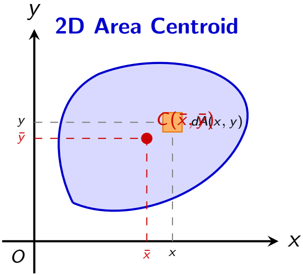
- **Conditions for Coincidence:**
  - When gravitational field g is uniform $\rightarrow$ CG coincides with CM.
  - When material density ρ is homogeneous $\rightarrow$ CM coincides with Centroid.

First Moment of Area (Q)

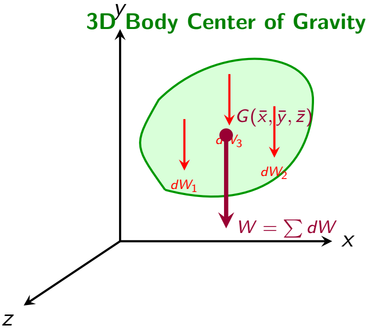
The first moment of a plane area A about a given axis is defined as the integral product of elemental area dA and its distance from that axis:

$$Q_y = \int x \, dA = \bar{x} \cdot A, \qquad Q_x = \int y \, dA = \bar{y} \cdot A$$

- **Centroid Coordinates for 2D Plane Area:**
$$\bar{x} = \frac{\int x \, dA}{\int dA} = \frac{\sum x_i A_i}{\sum A_i}, \qquad \bar{y} = \frac{\int y \, dA}{\int dA} = \frac{\sum y_i A_i}{\sum A_i}$$
- **Centre of Gravity Coordinates for 3D Mass Body:**
$$\bar{x} = \frac{\int x \, dW}{W} = \frac{\int x\rho \, dV}{\int \rho \, dV}, \quad \bar{y} = \frac{\int y \, dW}{W}, \quad \bar{z} = \frac{\int z \, dW}{W}$$
- **Dimensional Units:** First moment of area has units of mm^3 or m^3; centroid coordinates $\bar{x}, \bar{y}$ have units of length (mm or m).



2D Area Centroid
y
x
O
$C(\bar{x}, \bar{y})$
$dA(x, y)$
$\bar{y}$
$\bar{x}$



3D Body Center of Gravity
y
x
z
$G(\bar{x}, \bar{y}, \bar{z})$
dW_1
dW_2
dW_3
$W = \sum dW$

Symmetry Rules for Centroid Location

- **Single Axis of Symmetry:** If a plane section possesses an axis of symmetry, its centroid *must* lie along that axis.
- **Two Axes of Symmetry:** If a figure has two orthogonal axes of symmetry, the centroid is located at their intersection point.
- **Center of Symmetry:** For shapes with central symmetry (circle, square, regular hexagon), the point of symmetry is the centroid.

- **Symmetry Analysis Examples:**
  - **Rectangle ($b \times h$):** 2 axes of symmetry $\rightarrow C = (b/2, h/2)$.

- **Isosceles Triangle:** 1 axis of symmetry along median $\rightarrow C$ lies at $h/3$ from base.
- **Semicircle of Radius r:** 1 axis of symmetry along central vertical line $\rightarrow C$ lies at distance $\frac{4r}{3\pi}$ from base diameter.

Shape / Profile	**Area (A)**	$\bar{x}$ **(Ref Origin)**	$\bar{y}$ **(Ref Origin)**
Rectangle ($b \times h$)	$b \cdot h$	$b/2$	$h/2$
Right Triangle ($b \times h$)	$\frac{1}{2}bh$	$b/3$	$h/3$
Circle (radius r)	πr^2	r	r
Semicircle (radius r)	$\frac{\pi r^2}{2}$	0 (on sym. axis)	$\frac{4r}{3\pi} \approx 0.4244r$
Quarter Circle (radius r)	$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
Circular Sector (2α rad)	αr^2	$\frac{2r \sin \alpha}{3\alpha}$	0
Parabolic Spandrel ($b \times h$)	$\frac{1}{3}bh$	$\frac{3}{4}b$	$\frac{3}{10}h$

Table: Centroidal Properties of Standard 2D Geometric Shapes

Method of Segments (Composite Areas)

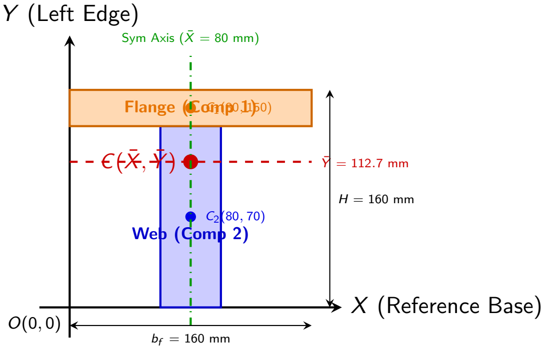
Structural shapes (I-beams, channels, T-sections, angles) are composed of combinations of simple standard shapes. The principle of moments dictates that the total area moment equals the algebraic sum of component moments.

- **Discrete Summation Equations:**

$$\bar{X} = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i} = \frac{A_1 x_1 + A_2 x_2 + \cdots + A_n x_n}{A_1 + A_2 + \cdots + A_n}$$

$$\bar{Y} = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i} = \frac{A_1 y_1 + A_2 y_2 + \cdots + A_n y_n}{A_1 + A_2 + \cdots + A_n}$$

- **Treatment of Holes and Cutouts:**
  - For subtracted areas (circular holes, rectangular cutouts, slots), assign a **negative sign** to area ($A_k < 0$).
  - The term $A_k x_k$ is subtracted in the numerator and A_k in the denominator.



Y (Left Edge)
Sym Axis ($\bar{X} = 80$ mm)
Flange (Comp 1)
$C_1(80, 150)$
$C(\bar{X}, \bar{Y})$
$\bar{Y} = 112.7$ mm
$H = 160$ mm
$C_2(80, 70)$
Web (Comp 2)
X (Reference Base)
$O(0, 0)$
$b_f = 160$ mm

Algorithm for Centroid Calculation

Follow this 6-step engineering methodology for analyzing any complex cross-section:

1. **Establish Reference Axes (X, Y):** Set origin $(0, 0)$ at the bottom-left corner or along an axis of symmetry.
2. **Decompose Section:** Divide complex shape into standard sub-shapes $(A_1, A_2, \ldots, A_n)$.
3. **Compute Sub-areas:** Calculate individual area A_i for each component (negative for cutouts).
4. **Locate Component Centroids:** Identify centroid coordinates (x_i, y_i) relative to the reference origin $(0, 0)$.
5. **Calculate Moments:** Evaluate $A_i \cdot x_i$ and $A_i \cdot y_i$ for every sub-shape.

6. **Apply Moment Theorem:** Compute overall centroid coordinates:

$$\bar{X} = \frac{\sum A_i x_i}{\sum A_i}, \qquad \bar{Y} = \frac{\sum A_i y_i}{\sum A_i}$$

Problem Statement: Locate centroid $(\bar{X}, \bar{Y})$ of an unequal angle section (120 mm $\times$ 100 mm $\times$ 10 mm) relative to bottom-left corner $O(0,0)$.

No.	**Component**	**Area A_i (mm^2)**	x_i **(mm)**	y_i **(mm)**	$A_i x_i$ **(mm^3)**	$A_i y_i$ **(mm^3)**
1	Horiz. Leg (100×10)	1000	50.0	5.0	$50,000$	$5,000$
2	Vert. Leg (10×110)	1100	5.0	65.0	$5,500$	$71,500$
$\sum$	**Total Section**	2100	–	–	$55,500$	$76,500$

Table: Tabular Centroid Calculation for Unequal Angle Section

- **Centroid Coordinates Computation:**

$$\bar{X} = \frac{\sum A_i x_i}{\sum A_i} = \frac{55,500 \text{ mm}^3}{2100 \text{ mm}^2} = \mathbf{26.43} \text{ mm}$$

$$\bar{Y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{76,500 \text{ mm}^3}{2100 \text{ mm}^2} = \mathbf{36.43} \text{ mm}$$

Structural Engineering Applications

- **Neutral Axis in Bending:** In beam bending mechanics ($\sigma = \frac{My}{I}$), the neutral axis passes directly through the centroid of the cross-section.
- **Shear Center Alignment:** Locating shear center relative to centroid prevents torsional twisting under lateral loads.

Mechanical & Vehicle Engineering Applications

- **Vehicle Rollover Stability:** Lower CG position increases static stability margin and reduces rollover propensity in high-speed cornering.
- **Dynamic Balancing:** Eccentricity of CG in high-speed rotors causes unwanted centrifugal vibration forces ($F = m\omega^2 e$).
- **Naval Hydrostatics:** Metacentric height ($GM = BM - BG$) dictates floatation stability of ships and submarines.

Core Concepts in Centroid Determination

Locating the centroid of a 2D plane area or the center of gravity of a 3D body requires defining a spatial reference frame. The selection of reference axes and identification of geometrical symmetry are fundamental steps in structural and mechanical design.

- **Centroid vs. Center of Gravity:**
  - *Centroid:* The point at which the total area of a 2D plane surface is assumed to be concentrated. It depends purely on geometry.
  - *Center of Gravity (CG):* The point through which the resultant gravitational force (weight) of a 3D body acts.
- **Role of Reference Axes:**
  - Coordinates $(\bar{x}, \bar{y})$ of a centroid are always measured relative to specified reference axes (OX and OY).

- Changing the reference axes alters numerical coordinate values but does not change the physical location of the centroid within the body.

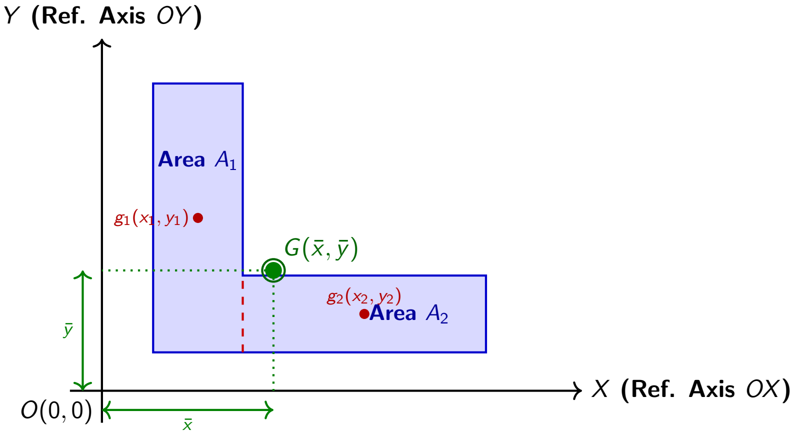
- **Role of Axis of Symmetry:**
  - Symmetry eliminates the need to calculate one or both centroidal coordinates.
  - If a plane figure possesses an axis of symmetry, its centroid always lies on that axis.

Definition of Reference Axes

The Axis of Reference consists of a set of mutually perpendicular lines (typically a horizontal axis OX and a vertical axis OY) from which the distances to various components of a plane area are measured.

- **Standard Reference Axis Conventions:**
  1. **Bottom and Left Boundaries (Default Rule):** The bottom-most horizontal line is chosen as the X-axis (OX), and the left-most vertical line is chosen as the Y-axis (OY). This keeps all coordinates positive ($\bar{x} > 0, \bar{y} > 0$).
  2. **Axis of Symmetry as Reference Axis:** If the section is symmetrical about a vertical line, that vertical line can be chosen as OY, yielding $\bar{x} = 0$.
  3. **Base of Component Shape:** Choosing the extreme bottom or extreme left edge of a composite figure simplifies distance measurements.

- **Mathematical Representation of Centroidal Distances:**
  - $\bar{x}$ = Perpendicular distance of centroid G from the vertical reference axis OY.
  - $\bar{y}$ = Perpendicular distance of centroid G from the horizontal reference axis OX.



Y (Ref. Axis OY)
Area A_1
$g_1(x_1, y_1)$
$G(\bar{x}, \bar{y})$
$g_2(x_2, y_2)$
Area A_2
$\bar{y}$
X (Ref. Axis OX)
$O(0, 0)$
$\bar{x}$

The Symmetry Theorem

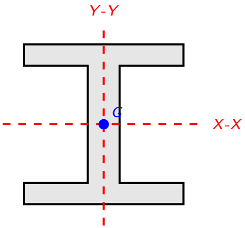
If a plane area possesses a line of symmetry, the centroid of the area MUST lie on that line of symmetry. If an area possesses two axes of symmetry, the centroid lies at their point of intersection.

- **Types of Symmetry in Engineering Sections:**
  1. **Dual Axis Symmetry (Symmetrical about both X-X and Y-Y):**
     - Examples: Rectangles, Circles, I-sections (Rolled Steel Beams - ISMB/ISLB), Ellipses.
     - Result: Centroid is located directly at the geometric center. No integration or area moment calculation is required!
  2. **Single Axis Symmetry (Symmetrical about one axis only):**
     - Symmetrical about Vertical Axis (Y-Y): T-sections, isosceles triangles, semi-circles resting horizontally. Here $\bar{x}$ is known by inspection ($\bar{x} = b/2$), only $\bar{y}$ must be computed.

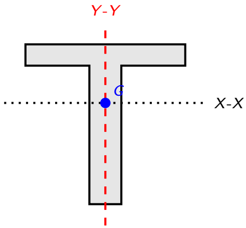
- Symmetrical about Horizontal Axis (X-X): Channel sections (ISMC), horizontal semi-circles. Here $\bar{y}$ is known by inspection ($\bar{y} = h/2$), only $\bar{x}$ must be computed.

3. **Asymmetrical Sections (No Axis of Symmetry):**
    - Examples: Unequal Angle sections (ISA), L-shapes, Scalene triangles, Z-sections.
    - Result: Both coordinates $\bar{x}$ and $\bar{y}$ must be calculated using moment equations.

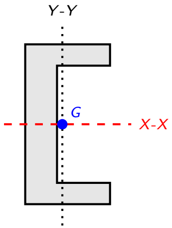
Visualizing Symmetry in Engineering Shapes I



(a) I-Section (Dual Axis)



(b) T-Section (Y-Axis Only)



(c) Channel (X-Axis Only)

Table: Centroidal Distances and Symmetry for Basic Geometric Shapes

Geometric Shape	**Parameters**	**Axis of Symmetry**	$\bar{x}$ **(from Ref. Axis)**	$\bar{y}$ **(from Ref. Axis)**	**Area** (A)
Rectangle	Width b, Depth d	Both X-X and Y-Y	$b/2$ (from left)	$d/2$ (from base)	$b \cdot d$
Triangle (Right-Angled)	Base b, Height h	None (Asymmetric)	$b/3$ (from vertical leg)	$h/3$ (from base)	$\frac{1}{2}bh$
Triangle (Isosceles)	Base b, Height h	Vertical Axis (Y-Y)	$b/2$ (from left)	$h/3$ (from base)	$\frac{1}{2}bh$
Circle	Radius R	All Diametral Axes	R (from left tangent)	R (from base tangent)	πR^2
Semicircle	Radius R (Base on X)	Vertical Axis (Y-Y)	R (from left tip)	$\frac{4R}{3\pi} \approx 0.424R$	$\frac{\pi R^2}{2}$
Quarter Circle	Radius R	Diagonal (45°) Line	$\frac{4R}{3\pi} \approx 0.424R$	$\frac{4R}{3\pi} \approx 0.424R$	$\frac{\pi R^2}{4}$
Parabolic Segment	Base b, Height h	Vertical Axis (Y-Y)	$b/2$ (from left)	$\frac{3}{5}h$ (from base)	$\frac{2}{3}bh$

Table: Real Engineering Data for Rolled Steel Sections (IS 808 / IS Handbook No. 1)

Section Type	**IS Designation**	**Flange b (mm)**	**Depth h (mm)**	**Symmetry Type**	**$\bar{x}$ (mm)**	**$\bar{y}$ (mm)**	**Area (A, cm^2)**
Beam (ISMB)	ISMB 200	100	200	Dual Axis (X&Y)	50.0	100.0	32.3
Beam (ISMB)	ISMB 300	140	300	Dual Axis (X&Y)	70.0	150.0	56.3
Channel (ISMC)	ISMC 150	50	150	Single (X-X only)	$c_x = 15.4$	75.0	20.8
Channel (ISMC)	ISMC 200	75	200	Single (X-X only)	$c_x = 21.7$	100.0	28.2
Equal Angle	ISA $75 \times 75 \times 6$	75	75	Diagonal ($c_x = c_y$)	20.6	20.6	8.63
Unequal Angle	ISA $100 \times 75 \times 8$	75	100	Asymmetric	18.7	31.0	13.4
Tee Section	ISNT 150	150	150	Single (Y-Y only)	75.0	$c_y = 41.8$	27.2

Principle of First Moments of Area

The moment of the total area about any reference axis is equal to the algebraic sum of the moments of its individual component areas about the same reference axis.

- **Mathematical Formulation:**

$$A \cdot \bar{x} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i \implies \bar{x} = \frac{\sum a_i x_i}{\sum a_i}$$

$$A \cdot \bar{y} = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n = \sum_{i=1}^{n} a_i y_i \implies \bar{y} = \frac{\sum a_i y_i}{\sum a_i}$$

- **5-Step Procedure for Locating Centroid of Composite Sections:**

1. **Inspect for Symmetry:** Check if the section is symmetrical about X-X or Y-Y. If symmetrical about Y-Y, set $\bar{x} =$ Half Width immediately.
2. **Establish Reference Axes:** Position OX along the lowest bottom line and OY along the extreme left boundary.
3. **Decompose Geometry:** Break complex section into simple shapes (rectangles, triangles, circles).
4. **Tabulate Component Data:** List Area (a_i), Centroidal Distances (x_i, y_i), and Area Moments ($a_i x_i, a_i y_i$).
5. **Calculate Overall Centroid:** Divide total area moment by total area.

Problem Statement

Find the centroid of a symmetrical T-section having a top flange of dimensions 120 mm $\times$ 10 mm and a vertical web of dimensions 10 mm $\times$ 100 mm.

- **Step 1: Symmetry Inspection & Reference Axes**
  - The section is symmetrical about the vertical axis Y-Y passing through the center of the web.
  - Therefore, $\bar{x} = \frac{120}{2} = 60$ mm from the extreme left edge of the flange.
  - Set Reference Axis OX along the bottom edge of the vertical web.
- **Step 2: Component Breakdown & Calculations**
  - *Component 1 (Vertical Web):*
    - Area $a_1 = 100 \text{ mm} \times 10 \text{ mm} = 1000 \text{ mm}^2$
    - Centroidal height $y_1 = \frac{100}{2} = 50$ mm from bottom reference OX.

- *Component 2 (Top Flange):*
  - Area $a_2 = 120 \text{ mm} \times 10 \text{ mm} = 1200 \text{ mm}^2$
  - Centroidal height $y_2 = 100 + \frac{10}{2} = 105$ mm from bottom reference OX.

- **Step 3: Total Area & Moment Summation**

$$\text{Total Area } A = a_1 + a_2 = 1000 + 1200 = 2200 \text{ mm}^2$$

$$\sum a_i y_i = (1000 \times 50) + (1200 \times 105) = 50000 + 126000 = 176000 \text{ mm}^3$$

- **Step 4: Final Centroid Location**

$$\bar{y} = \frac{\sum a_i y_i}{A} = \frac{176000}{2200} = 80 \text{ mm (from bottom edge of web)}$$

- **Conclusion:** Coordinates of Centroid G relative to bottom-left corner origin are $(60 \text{ mm}, 80 \text{ mm})$.

Concept of One-Dimensional (1-D) Geometrical Figures

A one-dimensional (1-D) element (such as a thin uniform wire, slender rod, curved filament, or reinforcement bar) is a body whose cross-sectional dimensions are negligible compared to its axial length L. Consequently, mass and geometric properties are analyzed strictly along the line geometry.

- **Centre of Gravity (G):** The point through which the total resultant gravitational force (weight $W = \int g\, dm$) acts, regardless of orientation.
- **Centroid ($\bar{G}$):** The geometric center of a figure. For a 1-D line element, it depends exclusively on the spatial distribution of length.
- **Equivalence Condition:** When density ρ and gravitational field g are constant throughout the wire:

$$\text{Centre of Gravity } (G) \equiv \text{Centroid } (\bar{G})$$

- **Principle of Moments (Varignon's Theorem for Lines):** The moment of the total wire length about any reference axis equals the algebraic sum of the moments of its constituent segments about the same axis:

$$\bar{x} \cdot L = \sum_{i=1}^{n} (l_i \cdot x_i) = \int x\, dl \implies \bar{x} = \frac{\sum_{i=1}^{n} l_i x_i}{\sum_{i=1}^{n} l_i}$$

$$\bar{y} \cdot L = \sum_{i=1}^{n} (l_i \cdot y_i) = \int y\, dl \implies \bar{y} = \frac{\sum_{i=1}^{n} l_i y_i}{\sum_{i=1}^{n} l_i}$$

Standard 1-D Wire Components & Coordinates

Reference formulas for fundamental 1-D line elements commonly encountered in structural frame and wire analysis.

Table: Standard 1-D Line Elements and Centroid Coordinates

Geometrical Shape	**Length (L)**	$\bar{x}$	$\bar{y}$	**Key Remarks / Symmetry**
Straight Wire Segment	$\sqrt{\Delta x^2 + \Delta y^2}$	$\frac{x_1+x_2}{2}$	$\frac{y_1+y_2}{2}$	Centroid at geometric midpoint
Circular Arc (Angle 2α)	$2R\alpha$	$\frac{R\sin\alpha}{\alpha}$	0	Symmetric about X-axis (α in rad)
Semicircular Wire Arc	πR	0	$\frac{2R}{\pi} \approx 0.6366R$	Symmetric about Y-axis
Quarter-Circular Wire Arc	$\frac{\pi R}{2}$	$\frac{2R}{\pi}$	$\frac{2R}{\pi} \approx 0.6366R$	Located in 1^{st} Quadrant
Line Segment along X-axis	L_0	$\frac{L_0}{2}$	0	Lies directly on reference axis

Derivation via First Principles (Integration)

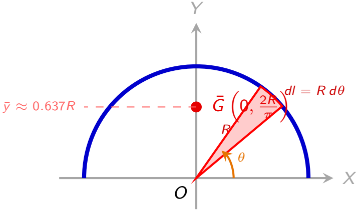
Deriving the centroid coordinate $\bar{y} = \frac{2R}{\pi}$ for a thin semicircular wire arc of radius R positioned symmetrically in the upper half-plane.

- **Differential Element Definition:** Consider a small differential element of arc length $dl = R\,d\theta$ subtending an angle $d\theta$ at polar angle $\theta \in [0, \pi]$.
- **Total Length of Semicircular Wire:**
$$L = \int_0^{\pi} dl = \int_0^{\pi} R\,d\theta = \pi R$$
- **First Moment of Length about X-axis (M_x):**
$$M_x = \int y\,dl = \int_0^{\pi} (R\sin\theta)(R\,d\theta) = R^2 \int_0^{\pi} \sin\theta\,d\theta$$
$$M_x = R^2[-\cos\theta]_0^{\pi} = R^2[-\cos(\pi) - (-\cos 0)] = R^2[1+1] = 2R^2$$

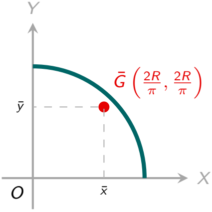
- **Applying Principle of Moments:**

$$\bar{y} \cdot L = M_x \implies \bar{y} \cdot (\pi R) = 2R^2 \implies \bar{y} = \frac{2R}{\pi} \approx 0.6366R$$

- **X-Coordinate:** By Y-axis symmetry, $\bar{x} = 0$.



(a) Semicircular Wire Arc



(b) Quarter-Circular Wire Arc

Systematic 5-Step Solution Procedure

Standard engineering protocol for determining the centroid of complex bent wire frames, rebar skeletons, and space trusses.

1. **Discretization of Wire Geometry:** Divide the continuous bent wire into n discrete, standard geometrical line segments (straight lines, semicircular or quarter-circular arcs).
2. **Reference Coordinate System Setup:** Select a suitable origin $O(0,0)$ and Cartesian axes $(X\text{-}Y)$. Prefer an extreme corner or hinge point to maintain positive coordinates.
3. **Individual Segment Parameter Evaluation:** For each segment i $(i = 1, 2, \ldots, n)$, evaluate:
    - Segment length l_i
    - Centroidal coordinates (x_i, y_i) relative to origin O

4. **First Moment Computation & Tabulation:** Compute the line moments $l_i x_i$ and $l_i y_i$ for each segment and calculate summations $\sum l_i$, $\sum l_i x_i$, and $\sum l_i y_i$.
5. **Composite Centroid Determination:** Apply the Principle of Moments:

$$\bar{x} = \frac{\sum_{i=1}^{n} l_i x_i}{\sum_{i=1}^{n} l_i}, \qquad \bar{y} = \frac{\sum_{i=1}^{n} l_i y_i}{\sum_{i=1}^{n} l_i}$$

Problem Statement (Structural Steel Wire Frame)

A uniform steel wire is bent into a planar frame consisting of three distinct segments:

- Segment 1 (AB): Horizontal wire along X-axis, length $l_1 = 120$ mm from $(0, 0)$ to $(120, 0)$.
- Segment 2 (BC): Vertical wire along Y-axis, length $l_2 = 180$ mm from $(120, 0)$ to $(120, 180)$.
- Segment 3 (CD): Inclined wire at 45° to horizontal, length $l_3 = 100$ mm from $(120, 180)$ to $(190.71, 250.71)$.

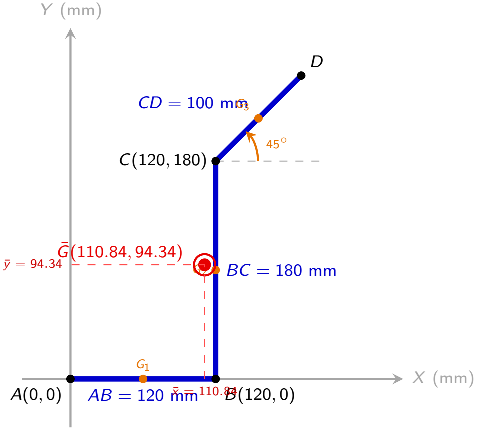
Calculate the overall centroid coordinates $(\bar{x}, \bar{y})$ of the wire frame.

Table: Tabular Centroidal Analysis for Problem 1

Segment	**Length l_i (mm)**	x_i **(mm)**	y_i **(mm)**	$l_i x_i$ **(mm^2)**	$l_i y_i$ **(mm^2)**
AB	120.0	60.0	0.0	7200.0	0.0
BC	180.0	120.0	90.0	21600.0	16200.0
CD	100.0	$120 + 50\cos 45^\circ = 155.36$	$180 + 50\sin 45^\circ = 215.36$	15535.5	21535.5
Total	$\sum l_i = 400.0$	–	–	$\sum l_i x_i = 44335.5$	$\sum l_i y_i = 37735.5$

Final Calculations

$$\bar{x} = \frac{44335.5}{400.0} = 110.84 \text{ mm}, \qquad \bar{y} = \frac{37735.5}{400.0} = 94.34 \text{ mm}$$



Y (mm)
X (mm)
D
CD = 100 mm
G_3
45°
C(120, 180)
$\bar{G}$(110.84, 94.34)
$\bar{y}$ = 94.34
BC = 180 mm
G_1
A(0, 0)
AB = 120 mm
$\bar{x}$ = 110.84
B(120, 0)

Problem Statement (Crane Hook / Curved Wire Member)

A thin structural tie wire consists of a straight segment AB of length 150 mm along the X-axis, seamlessly joined to a semicircular arc BC of radius $R = 80$ mm lying in the upper half-plane. Determine the centroid coordinates of the composite wire assembly.

Table: Tabular Centroidal Analysis for Problem 2

Segment	**Length l_i (mm)**	x_i **(mm)**	y_i **(mm)**	$l_i x_i$ **(mm^2)**	$l_i y_i$ **(mm^2)**
Straight AB	150.0	75.0	0.0	11250.0	0.0
Arc BC	$\pi \times 80 = 251.33$	$150 + 80 = 230.0$	$\frac{2\times80}{\pi} = 50.93$	57805.2	12800.0
Total	$\sum l_i = 401.33$	–	–	$\sum l_i x_i = 69055.2$	$\sum l_i y_i = 12800.0$

Composite Centroid Evaluation

Applying the Principle of Moments:

$$\bar{x} = \frac{\sum l_i x_i}{\sum l_i} = \frac{69055.2}{401.33} = 172.06 \text{ mm}$$

$$\bar{y} = \frac{\sum l_i y_i}{\sum l_i} = \frac{12800.0}{401.33} = 31.89 \text{ mm}$$

Observation: Note that $l_2 y_2 = (\pi R) \cdot \left(\frac{2R}{\pi}\right) = 2R^2 = 2(80)^2 = 12800.0\ \mathrm{mm}^2$, eliminating rounding errors during moment calculation.

Core Principles of 1-D Centroid Analysis

Key takeaways for engineering mechanics and structural design applications:

- **Line of Symmetry Rule:** If a 1-D wire figure possesses an axis of symmetry, its centroid $\bar{G}$ MUST lie on that axis. If two orthogonal axes of symmetry exist, $\bar{G}$ is located at their intersection.
- **External Centroid Location:** Unlike 2-D surfaces or 3-D solid bodies, the centroid of a curved 1-D wire (e.g., semicircular ring or bent bar) frequently lies in empty space outside the wire material itself.
- **Engineering Applications in Civil & Mechanical Design:**
  - Axis line determination for planar space frames and steel trusses to eliminate eccentric axial loading.

- Surface area calculation of solids of revolution using Theorem 1 of Pappus-Guldinus:

$$A = 2\pi \, \bar{y} \cdot L$$

- Routing and tension balancing of wire ropes, hoisting cables, and reinforcement steel bars (rebars).

Summary Equations

$$\bar{x} = \frac{\int x \, dl}{\int dl} = \frac{\sum l_i x_i}{\sum l_i}, \qquad \bar{y} = \frac{\int y \, dl}{\int dl} = \frac{\sum l_i y_i}{\sum l_i}$$

- **Course Context (DI03000201 – Engineering Mechanics)**:
  - In statics and structural design, understanding the spatial distribution of mass and area is critical for determining load resultants, support reactions, and structural equilibrium.
  - Centroid and Centre of Gravity provide single representative point locations for complex geometric shapes and distributed mass systems.
- **Fundamental Definitions**:
  1. **Centroid (G)**: The purely geometric centre of a plane figure (2D area) or volume (3D space). It depends solely on geometric shape and dimensions.
  2. **Centre of Gravity (CG)**: The point through which the single resultant force of gravity acts on a physical body, regardless of its spatial orientation.
  3. **Centre of Mass (CM)**: The point at which the entire mass of a rigid body may be assumed to be concentrated for dynamic analysis.

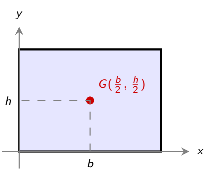
- **Equivalence Conditions**:
  - For homogeneous materials (density $\rho =$ constant) situated in a uniform gravitational field ($g =$ constant), the Centroid, Centre of Mass, and Centre of Gravity coincide identically ($G \equiv CM \equiv CG$).
- **First Moment Concepts**:
  - **First Moment of Area**: $Q_x = \int y\, dA = A\bar{y}$, $Q_y = \int x\, dA = A\bar{x}$.
  - **First Moment of Mass**: $M_{yz} = \int x\, dm = m\bar{x}$, $M_{xz} = \int y\, dm = m\bar{y}$, $\quad M_{xy} = \int z\, dm = m\bar{z}$.

- **Centroidal Coordinates for Elementary Shapes**:

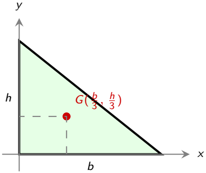
Plane Figure	**Area (A)**	$\bar{x}$	$\bar{y}$	**Reference Axis**
Square	a^2	$a/2$	$a/2$	Bottom-left corner
Rectangle	$b \cdot h$	$b/2$	$h/2$	Bottom-left corner
Right Triangle	$\frac{1}{2}bh$	$b/3$	$h/3$	Right-angled vertex
Circle	πr^2	r	r	Outer tangent corner
Semi-circle	$\frac{\pi r^2}{2}$	r	$\frac{4r}{3\pi} \approx 0.4244r$	Base center / diameter line
Quarter-circle	$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	Vertex of right angle

Table: Centroidal formulas and areas of standard two-dimensional plane shapes.

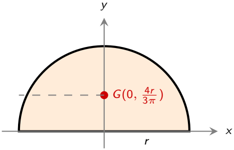
- **Symmetry Theorems for Centroids**:
  - If a plane figure possesses an axis of symmetry, its centroid always lies along that line of symmetry.
  - If a figure has two axes of symmetry (e.g., circle, rectangle, square), the centroid lies at their point of intersection.



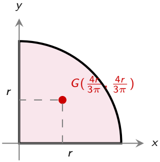
Rectangle



Right Triangle



Semi-circle



Quarter-circle

- **Principle of Superposition (First Moment of Area Method)**:
    - For complex structural shapes (I-sections, T-sections, bracket plates), the total area A is subdivided into N simple standard shapes ($N \leq 3$).
    - The overall centroid coordinates $(\bar{X}, \bar{Y})$ relative to chosen reference axes are computed using the first moment equations:

$$\bar{X} = \frac{\sum_{i=1}^{N} A_i \bar{x}_i}{\sum_{i=1}^{N} A_i} = \frac{A_1\bar{x}_1 + A_2\bar{x}_2 \pm A_3\bar{x}_3}{A_1 + A_2 \pm A_3}$$

$$\bar{Y} = \frac{\sum_{i=1}^{N} A_i \bar{y}_i}{\sum_{i=1}^{N} A_i} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2 \pm A_3\bar{y}_3}{A_1 + A_2 \pm A_3}$$

- **Treatment of Cutouts and Holes**:
    - Material subtractions (circular holes, triangular notches, cutouts) are assigned a **negative area** $(-A_k)$.
- **Systematic 5-Step Solution Procedure**:

1. Select a suitable origin and reference Cartesian coordinate axes $(X - Y)$.
2. Decompose the composite shape into simple components (rectangles, triangles, circles).
3. Calculate individual areas A_i and centroidal locations $(\bar{x}_i, \bar{y}_i)$.
4. Formulate tabular products $A_i\bar{x}_i$ and $A_i\bar{y}_i$.
5. Evaluate summations $\sum A_i$, $\sum A_i\bar{x}_i$, $\sum A_i\bar{y}_i$ and compute $(\bar{X}, \bar{Y})$.

- **Problem Statement**: A composite gusset plate consists of a rectangular base (120 mm $\times$ 80 mm), an attached right-angled triangle (60 mm base $\times$80 mm height on the right), and a circular hole ($\phi = 40$ mm, radius $r = 20$ mm) centered at (60 mm, 40 mm). Locate the overall centroid $(\bar{X}, \bar{Y})$ relative to the bottom-left corner of the rectangle.

Sub-figure (i)	**Area A_i (mm^2)**	$\bar{x}_i$ (mm)	$\bar{y}_i$ (mm)	$A_i\bar{x}_i$ (mm^3)	$A_i\bar{y}_i$ (mm^3)
1. Rectangle	$120 \times 80 = 9600$	60.00	40.00	576,000.0	384,000.0
2. Triangle	$\frac{1}{2} \times 60 \times 80 = 2400$	$120 + \frac{60}{3} = 140.00$	$\frac{80}{3} = 26.67$	336,000.0	64,008.0
3. Circle (Hole)	$-\pi(20)^2 = -1256.64$	60.00	40.00	−75,398.4	−50,265.6
Total ($\sum$)	10,743.36	–	–	836,601.6	397,742.4

Table: Tabular first moment of area calculations for the composite 2D plate.

- **Centroid Calculation**:

$$\bar{X} = \frac{\sum A_i \bar{x}_i}{\sum A_i} = \frac{836,601.6}{10,743.36} = \mathbf{77.87} \text{ mm}$$

$$\bar{Y} = \frac{\sum A_i \bar{y}_i}{\sum A_i} = \frac{397,742.4}{10,743.36} = \mathbf{37.02} \text{ mm}$$

- **First Moment of Mass for 3D Solids**:
  - For a 3D body of mass $M = \int dm = \int \rho \, dV$, the Centre of Gravity coordinates are:

$$\bar{x} = \frac{\int x \, dm}{M}, \quad \bar{y} = \frac{\int y \, dm}{M}, \quad \bar{z} = \frac{\int z \, dm}{M}$$

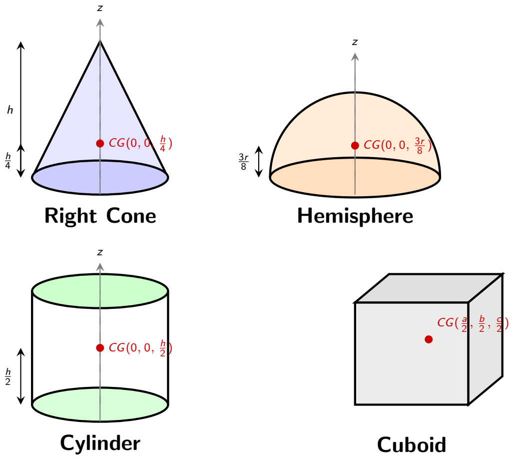
  - For homogeneous solids of constant density ρ, $dm = \rho \, dV$, reducing the mass moment equations to **First Moment of Volume**:

$$\bar{z} = \frac{\int z \, dV}{V} = \frac{\sum V_i \bar{z}_i}{\sum V_i}$$

3D Solid	Volume (V)	Centroidal Height ($\bar{z}$)	Reference Base
Cube (Side a)	a^3	$a/2$	Flat base
Cuboid ($a \times b \times c$)	$a \cdot b \cdot c$	$c/2$	Flat base ($a \times b$)
Cylinder (Radius r, Height h)	$\pi r^2 h$	$h/2$	Flat circular base
Cone (Radius r, Height h)	$\frac{1}{3}\pi r^2 h$	$h/4$	Flat circular base
Sphere (Radius r)	$\frac{4}{3}\pi r^3$	r	Center of sphere
Hemisphere (Radius r)	$\frac{2}{3}\pi r^3$	$\frac{3r}{8} = 0.375r$	Flat circular base

Table: Volume and Centre of Gravity equations for elementary 3D solids.

12.6 Centre of Gravity Locations for Standard 3D Solids I



- **Formulation for Multi-Component Solid Bodies**:
  - For composite solids composed of standard shapes (not exceeding two solids), the overall Centre of Gravity is determined by the first moment of mass:
$$\bar{Z}_{CG} = \frac{m_1 \bar{z}_1 \pm m_2 \bar{z}_2}{m_1 \pm m_2}$$
  - For uniform density materials ($\rho_1 = \rho_2$), mass is directly proportional to volume ($m_i = \rho V_i$), yielding:
$$\bar{Z}_{CG} = \frac{V_1 \bar{z}_1 \pm V_2 \bar{z}_2}{V_1 \pm V_2}$$
  - For bi-material composite components ($\rho_1 \neq \rho_2$):
$$\bar{Z}_{CG} = \frac{\rho_1 V_1 \bar{z}_1 + \rho_2 V_2 \bar{z}_2}{\rho_1 V_1 + \rho_2 V_2}$$
- **Additive vs. Subtractive Components**:
  - **Additive Solids**: Added masses (e.g., hemisphere attached to cylinder face) use positive volumes $+V_i$.
  - **Subtractive Solids**: Internal cavities (e.g., bored conical hole or spherical cavity) use negative volumes $-V_i$.

- **Problem Statement**: A steel machine component consists of a solid right circular cylinder of radius $R = 50$ mm and height $H = 120$ mm. A right circular conical cavity of base radius $R = 50$ mm and height $h = 60$ mm is bored out from the top flat face of the cylinder. Find the location of the Centre of Gravity ($\bar{Z}$) from the bottom base.

Solid Component (i)	**Volume V_i (mm^3)**	$\bar{z}_i$ (mm)	$V_i\bar{z}_i$ (mm^4)
1. Cylinder	$\pi(50)^2(120) = 300,000\pi$	$120/2 = 60.0$	$18,000,000\pi$
2. Conical Cavity (Bored)	$-\frac{1}{3}\pi(50)^2(60) = -50,000\pi$	$120 - \frac{60}{4} = 105.0$	$-5,250,000\pi$
Total ($\sum$)	$250,000\pi \approx 785,398.2$	–	$12,750,000\pi \approx 40,055,306$

Table: Volume and first moment calculation for cylinder with top conical cavity.

- **Calculation of Overall CG ($\bar{Z}$)**:

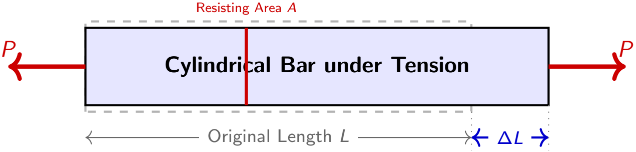
$$\bar{Z}_{CG} = \frac{\sum V_i\bar{z}_i}{\sum V_i} = \frac{12,750,000\pi}{250,000\pi} = \frac{12,750,000}{250,000} = \mathbf{51.00} \text{ mm}$$

- **Engineering Conclusion**: Removing top material shifts the Centre of Gravity downward from 60.0 mm to 51.00 mm, enhancing lower structural stability.

- **Definition of Direct (Axial) Stress (σ)**: The internal resisting force developed per unit cross-sectional area of a body perpendicular to the direction of an applied axial load.
$$\sigma = \frac{P}{A}$$
where P is the applied axial load (N or kN) and A is the original cross-sectional area (mm^2 or m^2).
- **Units of Stress**:
  - SI Base Unit: Pascal ($\text{Pa} = 1\ \text{N/m}^2$).
  - Engineering Units: Megapascal ($1\ \text{MPa} = 1\ \text{N/mm}^2 = 10^6\ \text{N/m}^2$) and Gigapascal ($1\ \text{GPa} = 10^3\ \text{N/mm}^2 = 10^9\ \text{N/m}^2$).
- **Definition of Linear (Axial) Strain (ϵ)**: The ratio of change in dimension (ΔL) along the load axis to the original dimension (L).

$$\epsilon = \frac{\Delta L}{L}$$

- **Nature of Strain**: Strain is a dimensionless scalar quantity often expressed as a percentage (%) or microstrain ($\mu\epsilon$).



- **Tensile Stress (σ_t)**:
  - Arises when two equal and opposite collinear axial pull forces act away from a section.
  - Induces elongation of the structural member ($\Delta L > 0$) accompanied by lateral contraction.
  - Standard Sign Convention: Positive ($+$).
- **Compressive Stress (σ_c)**:
  - Arises when two equal and opposite collinear axial push forces act toward a section.
  - Induces shortening of the structural member ($\Delta L < 0$) accompanied by lateral expansion.
  - Standard Sign Convention: Negative ($-$).
- **Governing Formulae**:

$$\sigma_t = +\frac{P}{A}, \quad \epsilon_t = +\frac{\Delta L}{L}$$
$$\sigma_c = -\frac{P}{A}, \quad \epsilon_c = -\frac{\Delta L}{L}$$

- **Elasticity**: The fundamental mechanical property of a material to completely recover its initial dimensions and shape upon removal of applied external deforming loads.
- **Plasticity**: The capacity of a material to undergo permanent, irreversible deformation without fracture when subjected to stresses exceeding its elastic boundary.
- **Elastic Limit**: The critical maximum stress intensity up to which a material behaves in a purely elastic manner.
  - If loaded below this limit, energy stored during loading is completely released upon unloading.
  - If loaded beyond this limit, a residual permanent set (ϵ_p) remains in the material.
- **Engineering Significance**: Design of mechanical components (shafts, bolts, pressure vessels) strictly restricts working stresses below the elastic limit to guarantee geometric stability.

- **Hooke's Law (Robert Hooke, 1676)**: Within the elastic limit of a material, direct stress is directly proportional to direct strain.

$$\sigma \propto \epsilon \implies \sigma = E \cdot \epsilon$$

- **Young's Modulus / Modulus of Elasticity (E)**: The constant ratio of direct stress to direct strain within the linear elastic range.

$$E = \frac{\sigma}{\epsilon} = \frac{P/A}{\Delta L/L} = \frac{P \cdot L}{A \cdot \Delta L}$$

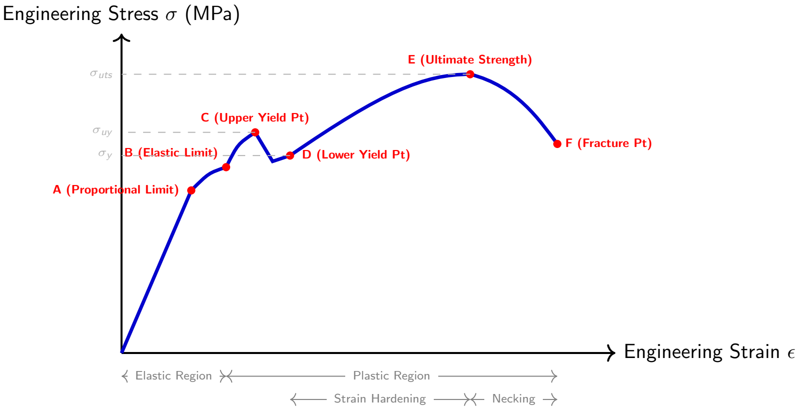
- **Derivation of Axial Elongation Formula**:

$$\Delta L = \frac{P \cdot L}{A \cdot E}$$

- **Physical Meaning of Young's Modulus**: Represents the inherent axial stiffness of a material. A higher value of E indicates higher resistance to elastic deformation under load.

Table: Typical Mechanical Properties of Structural Materials at 20°C

Material	**Young's Modulus** E (GPa)	**Yield Strength** σ_y (MPa)	**Ultimate Strength** σ_u (MPa)	**Poisson's Ratio** ν	**Elongation** (%)
Mild Steel (AISI 1020)	200	250	400	0.30	25%
Structural Steel (Fe 415)	205	415	540	0.29	20%
Stainless Steel (AISI 304)	193	215	505	0.29	40%
Aluminum Alloy (6061-T6)	68.9	276	310	0.33	12%
Copper (C11000)	117	70	220	0.34	45%
Gray Cast Iron (ASTM A48)	100	–	200	0.26	0.5%
Titanium Alloy (Ti-6Al-4V)	114	880	950	0.34	14%


Engineering Stress σ (MPa)
σuts
σuy
σy
E (Ultimate Strength)
C (Upper Yield Pt)
B (Elastic Limit)
D (Lower Yield Pt)
A (Proportional Limit)
F (Fracture Pt)
Engineering Strain ϵ
Elastic Region
Plastic Region
Strain Hardening
Necking

- **Proportional Limit (Point A)**: The maximum stress up to which the stress-strain curve remains a straight line $(O - A)$. Hooke's Law $(\sigma = E \cdot \epsilon)$ holds strictly true in this zone.
- **Elastic Limit (Point B)**: The limiting stress beyond which the material exhibits permanent plastic strain upon unloading.
- **Upper Yield Point (Point C)**: The stress point at which plastic slip initiates across crystal planes, causing sudden yielding without additional load.
- **Lower Yield Point (Point D)**: The stabilized stress value during plastic yield flow (σ_y). Serves as the primary structural design stress for ductile materials.
- **Ultimate Tensile Strength (Point E)**: The maximum stress value (σ_{uts}) based on the original cross-sectional area before localized necking begins.

- **Breaking / Rupture Point (Point F)**: Point of physical separation/fracture occurring at a reduced neck section under localized cup-and-cone shear failure.

- **Problem Statement**: A solid mild steel circular rod of length $L = 2.5$ m and uniform diameter $d = 20$ mm is subjected to a tensile axial force $P = 45$ kN. Given $E = 200$ GPa, calculate:
  1. Direct tensile stress (σ)
  2. Linear axial strain (ϵ)
  3. Total extension (ΔL) in mm.
- **Step-by-Step Solution**:
  - Cross-sectional Area (A):
  $$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(20)^2 = 314.16 \text{ mm}^2$$
  - Tensile Load (P): $P = 45 \text{ kN} = 45 \times 10^3$ N.
  - Direct Stress (σ):
  $$\sigma = \frac{P}{A} = \frac{45000 \text{ N}}{314.16 \text{ mm}^2} = 143.24 \text{ N/mm}^2 = 143.24 \text{ MPa}$$
  - Linear Strain (ϵ): ($E = 200 \text{ GPa} = 200 \times 10^3 \text{ N/mm}^2$)
  $$\epsilon = \frac{\sigma}{E} = \frac{143.24}{200000} = 7.162 \times 10^{-4} \quad \text{(or 0.0716\%)}$$

- Total Elongation (ΔL): ($L = 2.5$ m $= 2500$ mm)

$$\Delta L = \frac{P \cdot L}{A \cdot E} = (7.162 \times 10^{-4}) \times 2500 \text{ mm} = 1.791 \text{ mm}$$

- **Problem Statement**: A hollow structural steel column has an outer diameter $D = 100$ mm and an inner diameter $d = 80$ mm. The column carries an axial compressive load P. If the ultimate compressive strength of steel is $\sigma_u = 420$ MPa and a Factor of Safety (FOS) of 3.5 is specified, determine:
  1. Permissible working compressive stress (σ_{per})
  2. Maximum safe axial load (P_{safe}) in kN
  3. Compressive contraction (ΔL) over column length $L = 3.0$ m ($E = 205$ GPa).
- **Step-by-Step Solution**:
  - Permissible Stress (σ_{per}):

$$\sigma_{per} = \frac{\text{Ultimate Strength}}{\text{FOS}} = \frac{420 \text{ MPa}}{3.5} = 120.0 \text{ N/mm}^2$$

  - Cross-sectional Area (A):

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(100^2 - 80^2) = \frac{\pi}{4}(3600) = 2827.43 \text{ mm}^2$$

  - Maximum Safe Load (P_{safe}):

$$P_{safe} = \sigma_{per} \times A = 120.0 \times 2827.43 = 339{,}291.6 \text{ N} = 339.29 \text{ kN}$$

- Column Shortening (ΔL): ($L = 3000$ mm, $E = 205 \times 10^3$ N/mm^2)

$$\Delta L = \frac{\sigma_{per} \cdot L}{E} = \frac{120.0 \times 3000}{205000} = 1.756 \text{ mm}$$

- **Course Context**: GTU Course DI03000201 (Basics of Applied Mechanics), Unit: Direct Stress and Strain.
- **Focus Topics**:
    - Fundamentals of Direct Stress (Tensile and Compressive) and Linear Strain.
    - Physical concepts of Elasticity, Plasticity, and Elastic Limit.
    - Hooke's Law and Modulus of Elasticity (Young's Modulus E).
    - Experimental Stress-Strain curve for mild steel bar tested under uniaxial tension.
    - Numerical problem solving for stress, strain, elongation, and Young's modulus determination.
- **Engineering Importance**: Direct stress and linear strain form the foundational mechanics required for designing beams, columns, pressure vessels, and structural members under axial loads.

- **Definition**: Direct stress (σ) is defined as the internal resisting force per unit cross-sectional area offered by a body against deformation when subjected to an external axial load perpendicular (normal) to the cross-section.

- **Mathematical Equation**:

$$\sigma = \frac{P}{A} \tag{20}$$

- where P is the applied axial force (in N or kN) and A is the original cross-sectional area (in m^2 or mm^2).

- **Standard SI Units**:
    - Pascal (Pa $= 1$ N/m^2)
    - MegaPascal (1 MPa $= 10^6$ N/m^2 $= 1$ N/mm^2)
    - GigaPascal (1 GPa $= 10^9$ N/m^2 $= 10^3$ N/mm^2)

- **Classification**:
  - **Tensile Stress (σ_t)**: Developed when equal and opposite axial pulling forces stretch the member, increasing its length.
  - **Compressive Stress (σ_c)**: Developed when equal and opposite pushing forces crush the member, shortening its length.

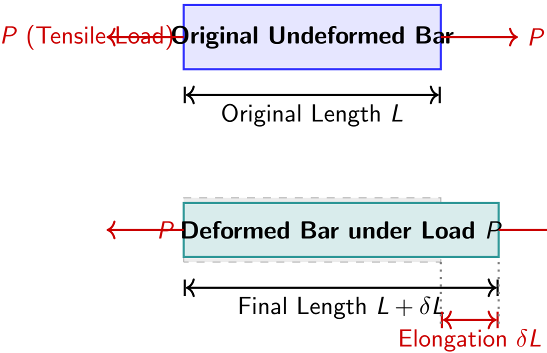
- **Definition**: Strain (ϵ) is the measure of deformation produced in a body due to external forces. Linear (or longitudinal/axial) strain is the ratio of the change in length to the original length measured along the line of action of the force.

- **Mathematical Formulation**:

$$\epsilon = \frac{\delta L}{L} \tag{21}$$

where δL is the change in length ($\delta L = L_{\text{final}} - L_{\text{original}}$) and L is the original length.

- **Properties of Linear Strain**:
  - **Dimensionless Quantity**: Since strain is the ratio of two length dimensions (mm/mm or m/m), it has no physical units.
  - Frequently expressed as a percentage (%) or in microstrain ($\mu\epsilon = 10^{-6}$).
  - **Tensile Strain**: Positive ($+\epsilon$), representing extension.
  - **Compressive Strain**: Negative ($-\epsilon$), representing contraction.



- **Kinematic Relationship**: Under axial force P, the member elongates by δL while experiencing minor lateral contraction (Poisson's effect).

- **Small Strain Assumption**: In elastic structural analysis, $\delta L \ll L$, allowing engineering calculations based on original area A and length L.

- **Elasticity**:
  - The mechanical property by virtue of which a material regains its original shape and dimensions completely upon the removal of external deforming forces.
  - Molecular mechanism: Interatomic bonds deform elastically like miniature springs without slipping.
- **Plasticity**:
  - The property by which a material retains permanent deformation after unloading (occurs when stresses exceed the elastic threshold).
- **Elastic Limit (σ_e)**:
  - The maximum value of direct stress up to which the material behaves perfectly elastically with zero residual permanent deformation after unloading.
  - Exceeding the elastic limit leads to irreversible plastic strain and permanent set.

- **Design Criterion**: Engineering structures (bridges, pressure vessels, machine shafts) are designed to operate strictly within the elastic limit to guarantee structural integrity.

- **Hooke's Law**: States that within the elastic limit (more precisely, up to the limit of proportionality), direct stress (σ) is directly proportional to linear strain (ϵ).

$$\sigma \propto \epsilon \implies \sigma = E \cdot \epsilon \tag{22}$$

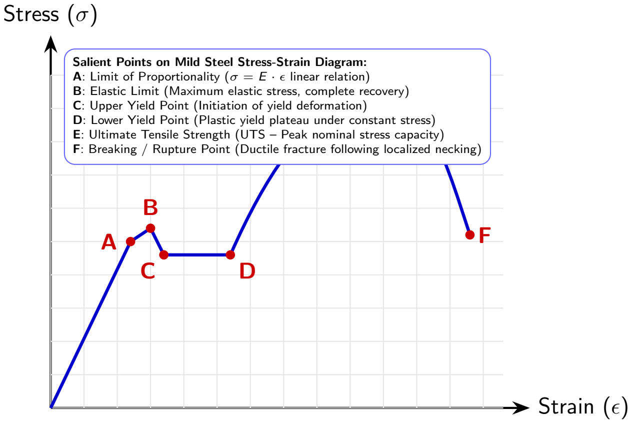
- **Young's Modulus / Modulus of Elasticity (E)**:

$$E = \frac{\sigma}{\epsilon} = \frac{P/A}{\delta L/L} = \frac{P \cdot L}{A \cdot \delta L} \tag{23}$$

E measures the stiffness (resistance to elastic deformation) of a solid material. Higher E indicates higher stiffness.

Table: Real Engineering Material Properties (Uniaxial Tension/Compression)

Engineering Material	**Young's Modulus** E (GPa)	**Yield Strength** σ_y (MPa)	**Ultimate Strength** σ_u (MPa)	**Poisson's Ratio** ν
Mild Steel (IS 2062)	200 – 210	250	410	0.28 – 0.30
Structural Steel (Fe 415)	200	415	540	0.30
Gray Cast Iron (FG 200)	100 – 120	–	200	0.21 – 0.26
Aluminum Alloy (6061-T6)	69 – 70	276	310	0.33
Copper Alloy (C11000)	110 – 120	70	220	0.34
Concrete (M25 Grade)	25	–	25 (Comp.)	0.15 – 0.20



- **Proportional Zone ($O - A$)**: Straight line graph; stress is directly proportional to strain.

- **Elastic Region ($A - B$)**: Slight non-linear curve up to elastic limit; zero permanent set.
- **Yielding Region ($B - C - D$)**: Strain increases rapidly without significant load increase due to dislocation slip.
- **Strain Hardening ($D - E$)**: Crystal structure rearranges, offering higher resistance up to UTS at point E.
- **Necking & Fracture ($E - F$)**: Cross section decreases locally (necking); nominal stress drops until ductile cup-and-cone fracture occurs at point F.

Problem Statement

A structural mild steel rod of length 2.5 m and circular cross-section diameter of 24 mm is subjected to an axial tensile force of 75 kN. Calculate:

1. Cross-sectional area of the rod (A),
2. Tensile stress developed (σ),
3. Tensile strain (ϵ), and
4. Total elongation (δL) of the rod.

Take Young's Modulus for mild steel
$E = 200 \text{ GPa} = 2 \times 10^5 \text{ N/mm}^2$.

Step-by-Step Solution:

- **Given Data**:
  - Length $L = 2.5 \text{ m} = 2500 \text{ mm}$

- Diameter $d = 24$ mm
- Load $P = 75$ kN $= 75 \times 10^3$ N
- Young's Modulus $E = 200$ GPa $= 2 \times 10^5$ N/mm^2

- **Step 1: Cross-Sectional Area (A)**:

$$A = \frac{\pi}{4} \cdot d^2 = \frac{\pi}{4} \cdot (24)^2 = 452.39 \text{ mm}^2 = 4.5239 \times 10^{-4} \text{ m}^2 \tag{24}$$

- **Step 2: Tensile Stress (σ)**:

$$\sigma = \frac{P}{A} = \frac{75 \times 10^3 \text{ N}}{452.39 \text{ mm}^2} = 165.79 \text{ N/mm}^2 = 165.79 \text{ MPa} \tag{25}$$

- **Step 3: Linear Tensile Strain (ϵ)**:

$$\epsilon = \frac{\sigma}{E} = \frac{165.79 \text{ N/mm}^2}{2 \times 10^5 \text{ N/mm}^2} = 8.2895 \times 10^{-4} = 0.0829\% \tag{26}$$

- **Step 4: Total Elongation (δL)**:

$$\delta L = \epsilon \cdot L = (8.2895 \times 10^{-4}) \times 2500 \text{ mm} = 2.072 \text{ mm} \tag{27}$$

Alternatively, using direct axial deformation formula:

$$\delta L = \frac{P \cdot L}{A \cdot E} = \frac{(75000) \times (2500)}{(452.39) \times (200000)} = 2.072 \text{ mm} \tag{28}$$

Problem Statement

During a standard tensile test conducted on a mild steel specimen using a Universal Testing Machine (UTM), the following experimental observation data was recorded within the elastic limit:

- Specimen original diameter $(d) = 12.5$ mm
- Gauge length $(L) = 200$ mm
- Applied axial tensile load $(P) = 24$ kN
- Extension measured over gauge length $(\delta L) = 0.21$ mm

Determine the Young's Modulus (E) of the test material in GPa.

Step-by-Step Solution:

- **Step 1: Calculate Specimen Area (A)**:

$$A = \frac{\pi}{4} \cdot d^2 = \frac{\pi}{4} \cdot (12.5)^2 = 122.72 \text{ mm}^2 \tag{29}$$

- **Step 2: Calculate Direct Stress (σ)**:

$$\sigma = \frac{P}{A} = \frac{24 \times 10^3 \text{ N}}{122.72 \text{ mm}^2} = 195.57 \text{ N/mm}^2 = 195.57 \text{ MPa} \tag{30}$$

- **Step 3: Calculate Axial Strain (ϵ)**:

$$\epsilon = \frac{\delta L}{L} = \frac{0.21 \text{ mm}}{200 \text{ mm}} = 1.05 \times 10^{-3} \tag{31}$$

- **Step 4: Calculate Young's Modulus (E)**:

$$E = \frac{\sigma}{\epsilon} = \frac{195.57 \text{ N/mm}^2}{1.05 \times 10^{-3}} = 186257.14 \text{ N/mm}^2 = 186.26 \text{ GPa} \tag{32}$$

- **Engineering Conclusion**: The calculated modulus (186.26 GPa) is within typical elastic range for commercial structural steels, accounting for experimental gauge tolerance.

Table: Summary of Core Mathematical Relations in Direct Stress and Strain

Parameter / Law	Formula	Standard Units
Direct Stress	$\sigma = \frac{P}{A}$	N/mm^2 (MPa) or N/m^2 (Pa)
Linear Strain	$\epsilon = \frac{\delta L}{L}$	Dimensionless (%, $\mu\epsilon$)
Hooke's Law	$\sigma = E \cdot \epsilon$	N/mm^2 (for E and σ)
Young's Modulus	$E = \frac{\sigma}{\epsilon}$	GPa (1 GPa $= 10^3$ N/mm^2)
Axial Deformation	$\delta L = \frac{P \cdot L}{A \cdot E}$	mm or m

- **Key Takeaways**:
  - Direct stress measures internal load intensity; linear strain measures intensity of deformation along load axis.
  - Hooke's Law governs linear elastic material response up to proportionality limit.

- Mild steel stress-strain diagram exhibits distinct phases: Elastic extension $(O - B)$, Yielding $(C - D)$, Strain hardening $(D - E)$, and Ductile necking/rupture $(E - F)$.
- Safe structural component design enforces working stresses well below the elastic limit.

- **Course & Unit Context**: GTU Course DI03000201 (Mechanics of Solids / Strength of Materials), Unit: Direct Stress and Strain.
- **Focus Topics**:
  - Lateral strain, lateral stress, and Poisson's ratio (ν).
  - Volumetric strain (ϵ_v) under uniaxial, biaxial, and triaxial loading states.
  - Bulk modulus (K) and hydrostatic stress representation.
  - Inter-relationships between elastic constants (E, G, K, ν).
  - Fundamental concepts of shear stress (τ), shear strain (γ), and modulus of rigidity (G).
  - Step-by-step solved numerical problem with complete parametric evaluation.
- **Learning Objectives**:
  - Differentiate between longitudinal and lateral strain under axial loading.

- Calculate volumetric deformation for solids under multidirectional stress fields.
- Derive and apply the mathematical relationships connecting E, G, K, and ν.
- Analyze shear stress action and compute rigidity modulus from experimental deformation data.

- **Longitudinal Strain (ϵ_{long})**:
  - Strain measured along the line of action of the applied axial force.
  - $\epsilon_x = \frac{\delta L}{L}$ (tensile is positive, compressive is negative).
- **Lateral Strain (ϵ_{lat})**:
  - Strain occurring perpendicular to the line of action of the applied load.
  - Under axial tension along X-axis, lateral dimensions shrink:

$$\epsilon_y = \frac{\delta b}{b}, \quad \epsilon_z = \frac{\delta d}{d} \tag{33}$$

  - Lateral strain always possesses the **opposite sign** to longitudinal strain.

- **Poisson's Ratio (ν or μ or $1/m$)**:
  - Defined as the ratio of lateral strain to longitudinal strain within the elastic limit:

$$\nu = -\frac{\text{Lateral Strain}}{\text{Longitudinal Strain}} = \frac{|\epsilon_{\text{lateral}}|}{\epsilon_{\text{longitudinal}}} \tag{34}$$

  - For an axially loaded bar along X-axis: $\epsilon_y = \epsilon_z = -\nu\epsilon_x = -\nu\frac{\sigma_x}{E}$.
- **Physical Limits**:
  - Theoretical range: $-1.0 \leq \nu \leq 0.5$.
  - Engineering solids: $0.0 \leq \nu \leq 0.50$.
  - Incompressible materials (e.g., rubber): $\nu \approx 0.50$.
  - Zero lateral deformation materials (e.g., cork): $\nu \approx 0.0$.

Table: Elastic Constants and Poisson's Ratio for Common Engineering Materials at 20°C

Material	Young's Modulus E (GPa)	Shear Modulus G (GPa)	Bulk Modulus K (GPa)	Poisson's Ratio ν
Mild Steel (IS 2062)	200 – 210	78 – 82	160 – 170	0.28 – 0.30
Structural Steel (Fe410)	205	80	164	0.29
Aluminum Alloy (6061-T6)	69 – 70	25 – 26	70 – 75	0.33
Copper (Commercial Pure)	110 – 120	40 – 48	120 – 140	0.34
Brass (70/30 Cartridge)	100 – 110	37 – 40	105 – 115	0.35
Grey Cast Iron	100 – 130	40 – 50	90 – 110	0.23 – 0.27
Titanium Alloy (Ti-6Al-4V)	110 – 114	42 – 45	110 – 115	0.32
Concrete (M25 Grade)	25 – 30	10 – 12	14 – 18	0.15 – 0.20
Rubber (Vulcanized Natural)	0.01 – 0.10	0.0003 – 0.003	1.5 – 2.0	0.48 – 0.499
Cork	0.005 – 0.010	0.002 – 0.004	0.005 – 0.010	0.00

- **Key Observation**: For standard structural metals, $G \approx 0.38E$ to $0.40E$, and $K \approx 0.8E$ to $1.2E$.
- As $\nu \to 0.50$ (incompressible behavior like rubber), bulk modulus $K \to \infty$.

- **Volumetric Strain (ϵ_v)**:
  - Defined as the fractional change in volume of a body subjected to stress:

$$\epsilon_v = \frac{\delta V}{V} \tag{35}$$

- **Uniaxial Axial Loading**:

- For a bar of length L, width b, and depth d under tensile stress σ_x:

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z = \epsilon_x - \nu\epsilon_x - \nu\epsilon_x = \epsilon_x(1-2\nu) = \frac{\sigma_x}{E}(1-2\nu) \tag{36}$$

- **Triaxial Stress Field**:
  - For perpendicular stresses $\sigma_x, \sigma_y, \sigma_z$:

$$\epsilon_v = \frac{\sigma_x + \sigma_y + \sigma_z}{E}(1-2\nu) \tag{37}$$

- **Hydrostatic State of Stress & Bulk Modulus (K)**:

- When uniform compressive or tensile pressure p acts in all directions ($\sigma_x = \sigma_y = \sigma_z = -p$):

$$\epsilon_V = -\frac{3p}{E}(1 - 2\nu) \tag{38}$$

- **Bulk Modulus (K)** is defined as the ratio of hydrostatic stress to volumetric strain:

$$K = \frac{p}{|\epsilon_V|} = \frac{E}{3(1 - 2\nu)} \tag{39}$$

- **The Four Basic Elastic Constants**:
  - Young's Modulus of Elasticity (E)
  - Shear Modulus / Modulus of Rigidity (G or C)
  - Bulk Modulus (K)
  - Poisson's Ratio (ν)
- **Derivation 1: Relation between** $E, G,$ **and** ν:

$$E = 2G(1+\nu) \quad \Longrightarrow \quad \nu = \frac{E}{2G} - 1 \tag{40}$$

- **Derivation 2: Relation between** $E, K,$ **and** ν:

$$E = 3K(1-2\nu) \quad \Longrightarrow \quad \nu = \frac{1}{2}\left(1 - \frac{E}{3K}\right) \tag{41}$$

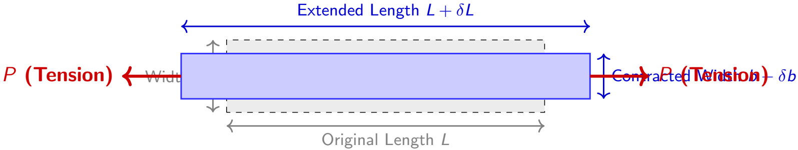
- **Derivation 3: Relation combining E, G, and K:**
    - Equating expressions for ν from Derivations 1 and 2:

$$\frac{E}{2G} - 1 = \frac{1}{2}\left(1 - \frac{E}{3K}\right) \implies E = \frac{9KG}{3K + G} \tag{42}$$

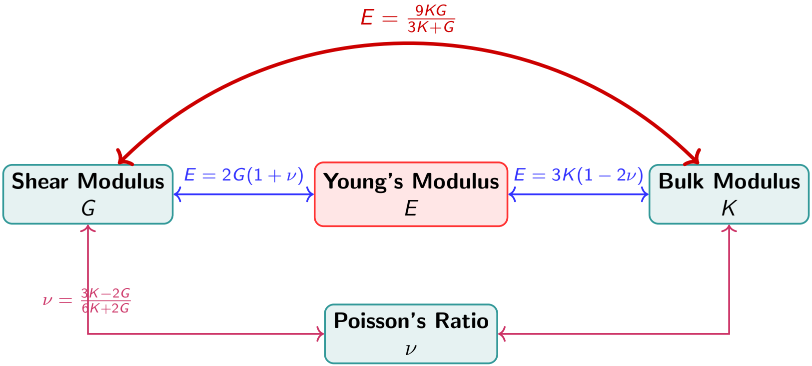
- **Derivation 4: Poisson's ratio in terms of K and G:**

$$\nu = \frac{3K - 2G}{6K + 2G} \tag{43}$$

1. Lateral Contraction Under Tensile Loading:



2. Inter-relationship Network of Elastic Constants:


$E = \frac{9KG}{3K+G}$
Shear Modulus
G
$E = 2G(1+\nu)$
Young's Modulus
E
$E = 3K(1-2\nu)$
Bulk Modulus
K
$\nu = \frac{3K-2G}{6K+2G}$
Poisson's Ratio
ν

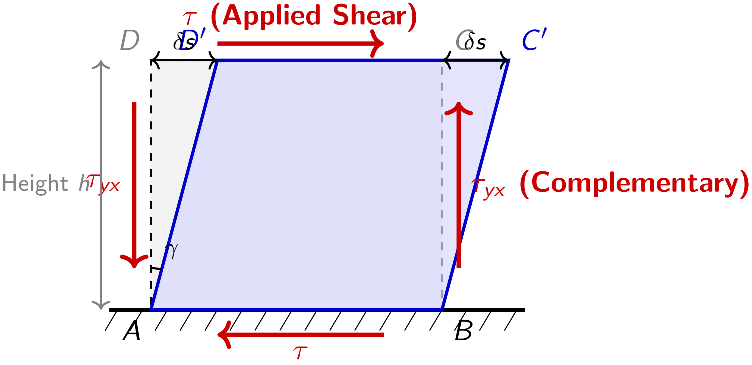
- **Shear Stress (τ)**:
  - Stress produced when an applied force acts parallel or tangential to the cross-sectional plane.
  - $$\tau = \frac{\text{Tangential Shear Force } P_s}{\text{Resisting Area } A} \quad [\text{N/mm}^2 \text{ or MPa}] \tag{44}$$
- **Complementary Shear Stress Theorem**:
  - A single shear stress cannot exist in equilibrium.
  - Shear stress on any plane is always accompanied by an equal and opposite complementary shear stress on the perpendicular plane ($\tau_{xy} = \tau_{yx}$).
- **Shear Strain (γ)**:
  - Angular distortion (in radians) produced by shear stress between two parallel faces.

- For a block of height h shifted laterally by δs:

$$\gamma = \tan\gamma \approx \frac{\delta s}{h} \quad \text{[radians]} \tag{45}$$

- **Modulus of Rigidity / Shear Modulus (**G **or** C **or** N**)**:
  - By Hooke's Law for shear within elastic limit:

$$\tau \propto \gamma \implies \tau = G \cdot \gamma \implies G = \frac{\tau}{\gamma} \quad \text{[GPa or MPa]} \tag{46}$$



τ (Applied Shear)
D
δs
D'
C
δs
C'
Height h
τ_{yx}
τ_{yx} (Complementary)
γ
A
B
τ

Problem Statement: A cylindrical steel metallic bar of diameter $d = 30$ mm and original length $L = 300$ mm is subjected to an axial tensile force $P = 120$ kN. Under this load, the extension in length is measured as $\delta L = 0.25$ mm and the decrease in diameter is $\delta d = 0.0075$ mm.

Determine:

1. Longitudinal strain (ϵ_x) and Lateral strain (ϵ_y).
2. Poisson's ratio (ν).
3. Young's Modulus of Elasticity (E).
4. Shear Modulus / Modulus of Rigidity (G).
5. Bulk Modulus (K).
6. Volumetric strain (ϵ_v) and total change in volume (δV).

Step 1: Cross-Sectional Area and Initial Volume:

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(30)^2 = 706.86 \text{ mm}^2 \tag{47}$$

$$V = A \times L = 706.86 \times 300 = 212,058 \text{ mm}^3 \tag{48}$$

Step 2: Strains and Poisson's Ratio:

$$\epsilon_x = \frac{\delta L}{L} = \frac{0.25}{300} = 8.333 \times 10^{-4} \quad (0.08333\%) \tag{49}$$

$$\epsilon_y = \frac{\delta d}{d} = \frac{0.0075}{30} = 2.500 \times 10^{-4} \quad (0.02500\%) \tag{50}$$

$$\nu = \frac{\epsilon_y}{\epsilon_x} = \frac{2.500 \times 10^{-4}}{8.333 \times 10^{-4}} = 0.30 \tag{51}$$

Step 3: Axial Stress and Young's Modulus (E):

$$\sigma_x = \frac{P}{A} = \frac{120 \times 10^3 \text{ N}}{706.86 \text{ mm}^2} = 169.76 \text{ N/mm}^2 = 169.76 \text{ MPa} \tag{52}$$

$$E = \frac{\sigma_x}{\epsilon_x} = \frac{169.76}{8.333 \times 10^{-4}} = 203,720 \text{ N/mm}^2 = 203.72 \text{ GPa} \tag{53}$$

Step 4: Shear Modulus (G):

$$E = 2G(1+\nu) \implies G = \frac{E}{2(1+\nu)} = \frac{203.72}{2(1+0.30)} = 78.35 \text{ GPa} \tag{54}$$

Step 5: Bulk Modulus (K):

$$E = 3K(1-2\nu) \implies K = \frac{E}{3(1-2\nu)} = \frac{203.72}{3(1-2(0.30))} = \frac{203.72}{1.20} = 169.77 \text{ GPa} \tag{55}$$

Step 6: Volumetric Strain (ϵ_v) and Volume Change (δV):

$$\epsilon_v = \epsilon_x(1-2\nu) = (8.333\times10^{-4})(1-2(0.30)) = 3.333\times10^{-4} \tag{56}$$

$$\delta V = V \times \epsilon_v = 212,058 \times (3.333\times10^{-4}) = 70.686 \text{ mm}^3 \tag{57}$$

- **Core Elastic Formulae Quick-Reference**:

Quantity	Governing Equation
Poisson's Ratio (ν)	$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}} = \frac{\delta d/d}{\delta L/L}$
Volumetric Strain (ϵ_v)	$\epsilon_v = \frac{\delta V}{V} = \epsilon_x + \epsilon_y + \epsilon_z = \frac{\sigma_x+\sigma_y+\sigma_z}{E}(1-2\nu)$
Bulk Modulus (K)	$K = \frac{\text{Direct Stress}}{\text{Volumetric Strain}} = \frac{E}{3(1-2\nu)}$
Modulus of Rigidity (G)	$G = \frac{\text{Shear Stress } \tau}{\text{Shear Strain } \gamma} = \frac{E}{2(1+\nu)}$
Master Relation (E, G, K)	$E = \frac{9KG}{3K+G}$

- **Key Exam Takeaways**:
  - Isotropic elastic solids require only **two independent** elastic constants to fully define stress-strain behavior.
  - For incompressible materials ($\nu = 0.5$), volumetric strain $\epsilon_v = 0$ and $K \to \infty$.
  - Shear stress is always accompanied by an equal complementary shear stress on mutually perpendicular planes.

- **Course & Context:** GTU Course DI03000201 (Direct Stress and Strain), Lecture 16.
- **Definition of Composite / Compound Section:**
  - A structural member constructed by rigidly bonding two or more different materials (e.g., steel and copper, steel and concrete, brass and aluminum) together along their length.
  - Under an applied external load, all constituent materials deform as a single integrated monolithic unit.
- **Fundamental Assumptions in Analysis:**
  - *Perfect Bonding:* No relative slipping or shear sliding occurs at the interface between constituent materials.
  - *Equal Strain / Compatibility:* The axial deformation (δL) and strain (ε) experienced by all constituent materials are equal.
  - *Elastic Behavior:* All materials obey Hooke's Law ($\sigma = E \cdot \varepsilon$) within their proportional limits.
- **Engineering Applications:**

- Reinforced Cement Concrete (RCC) columns and beams (steel rebars encased in concrete).
- Steel-encased concrete columns, bimetallic thermal strips, flitched timber beams, and composite drive shafts.

- **Strain Compatibility Condition:**
  - Since materials are rigidly joined at the end plates or interface:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon$$

  - Applying Hooke's Law ($\varepsilon = \sigma/E$):

$$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \implies \sigma_1 = \left(\frac{E_1}{E_2}\right)\sigma_2$$

- **Modular Ratio (m):**
  - The ratio of the Modulus of Elasticity of Material 1 to that of Material 2:

$$m = \frac{E_1}{E_2}$$

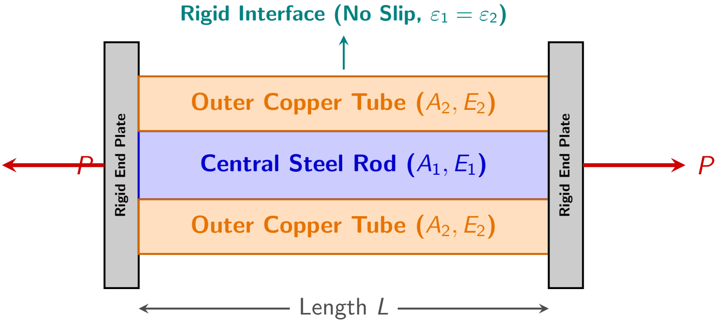
  - Direct Stress Relationship: $\sigma_1 = m \cdot \sigma_2$.
  - Stiffer material (higher E) carries proportionally higher axial stress!
- **Load Equilibrium Condition:**
  - Total external axial force P equals the sum of loads resisted by component 1 (P_1) and component 2 (P_2):

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$

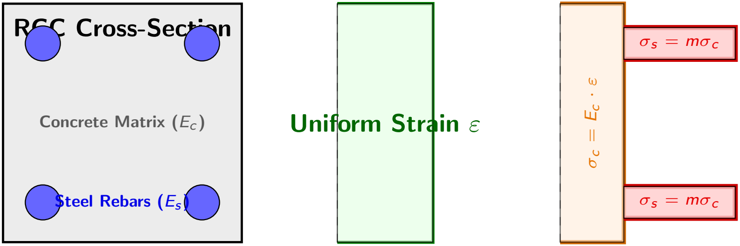
- Substituting $\sigma_1 = m\sigma_2$:

$$P = (m\sigma_2)A_1 + \sigma_2 A_2 = \sigma_2 (mA_1 + A_2)$$

- Equivalent Area (A_{eq}): $A_{eq} = mA_1 + A_2$ (Transformed area in terms of Material 2).



Rigid Interface (No Slip, $\varepsilon_1 = \varepsilon_2$)
Outer Copper Tube (A_2, E_2)
Central Steel Rod (A_1, E_1)
Outer Copper Tube (A_2, E_2)
Rigid End Plate
Rigid End Plate
P
P
Length L



$\varepsilon_s = \varepsilon_c = \varepsilon$

Stress Jump ($\sigma_s = m \cdot \sigma_c$)

Engineering Material	**Young's Modulus E (GPa)**	**Modular Ratio m (E/E_{concrete})**	**Yield Strength f_y (MPa)**	**Thermal Coeff. α (10^{-6}/K)**
Structural Steel (Fe 415 / Fe 500)	200	8.00	415 – 500	12.0
Copper (C11000 Annealed)	110 – 120	4.40 – 4.80	70 – 220	16.8
Aluminum Alloy (6061-T6)	69 – 71	2.76 – 2.84	240 – 276	23.1
Structural Brass (C36000)	100 – 105	4.00 – 4.20	180 – 310	20.5
Structural Concrete (M25 Grade)	25 ($E_c = 5000\sqrt{f_{ck}}$)	1.00 (Reference)	25 (f_{ck})	10.0
Structural Timber (Teak Wood)	11 – 14	0.44 – 0.56	35 – 60	5.0

- **Engineering Significance of Table Data:**
  - Modular ratio m quantifies the relative stiffness of composite materials.
  - Steel has $m = 8.00$ relative to M25 concrete; thus, steel carries 8 times the stress of surrounding concrete at equal strain.

1. **Step 1: Determine Component Cross-Sectional Areas**
   - Solid bar: $A = \frac{\pi}{4} d^2$.
   - Hollow tube / Sleeve: $A = \frac{\pi}{4}(D_{outer}^2 - d_{inner}^2)$.
   - Rebars in matrix: $A_s = n \cdot \frac{\pi}{4} d_b^2$, $A_{matrix} = A_{gross} - A_s$.
2. **Step 2: Calculate Modular Ratio**
   - Compute $m = \frac{E_1}{E_2}$ (where Material 1 is stiffer, $E_1 > E_2$).
3. **Step 3: Apply Strain Compatibility Equation**
   - Express stress in Material 1 in terms of Material 2: $\sigma_1 = m \cdot \sigma_2$.
4. **Step 4: Set Up Load Equilibrium Equation**
   - Substitute σ_1 into $P = \sigma_1 A_1 + \sigma_2 A_2$:

$$P = \sigma_2(mA_1 + A_2) \implies \sigma_2 = \frac{P}{mA_1 + A_2}$$

5. **Step 5: Compute Stresses, Component Loads & Deformation**
   - $\sigma_1 = m \cdot \sigma_2$, $P_1 = \sigma_1 A_1$, $P_2 = \sigma_2 A_2$.
   - Total elongation $\delta L = \frac{\sigma_1 L}{E_1} = \frac{\sigma_2 L}{E_2}$.

Problem Statement

A compound bar of length $L = 500$ mm consists of a solid central steel rod of diameter $d_s = 25$ mm enclosed rigidly inside a copper tube of internal diameter 25 mm and external diameter $d_c = 40$ mm. The assembly is subjected to an axial tensile load $P = 120$ kN.

Given: $E_s = 200$ GPa $= 200 \times 10^3$ N/mm^2,
$E_c = 100$ GPa $= 100 \times 10^3$ N/mm^2.

Calculate:

1. Modular ratio $m = E_s/E_c$.
2. Induced tensile stresses in steel (σ_s) and copper (σ_c).
3. Tensile load carried by steel (P_s) and copper (P_c).
4. Total elongation of the compound bar (δL).

- **Step 1: Cross-Sectional Areas**

$$A_s = \frac{\pi}{4}(25)^2 = 490.87 \text{ mm}^2$$

$$A_c = \frac{\pi}{4}(40^2 - 25^2) = \frac{\pi}{4}(1600 - 625) = 765.76 \text{ mm}^2$$

- **Step 2: Modular Ratio & Stress Compatibility**

$$m = \frac{E_s}{E_c} = \frac{200 \times 10^3}{100 \times 10^3} = 2.0 \implies \sigma_s = 2.0 \cdot \sigma_c$$

- **Step 3: Load Equilibrium & Stress Evaluation**

$$P = \sigma_s A_s + \sigma_c A_c = (2\sigma_c)(490.87) + \sigma_c(765.76) = 1747.50 \cdot \sigma_c$$

$$120 \times 10^3 \text{ N} = 1747.50 \cdot \sigma_c \implies \sigma_c = 68.67 \text{ N/mm}^2 \text{ (MPa)}$$

$$\sigma_s = 2.0 \times 68.67 = 137.34 \text{ N/mm}^2 \text{ (MPa)}$$

- **Step 4: Load Distribution**

$$P_s = \sigma_s A_s = 137.34 \times 490.87 = 67,416 \text{ N} = 67.42 \text{ kN}$$

$$P_c = \sigma_c A_c = 68.67 \times 765.76 = 52,584 \text{ N} = 52.58 \text{ kN} \quad (\text{Total } P = 120 \text{ kN})$$

- **Step 5: Total Elongation**

$$\delta L = \frac{\sigma_s L}{E_s} = \frac{137.34 \times 500}{200 \times 10^3} = 0.343 \text{ mm}$$

Problem Statement & Solution

A short square RCC column (300 mm $\times$ 300 mm) is reinforced with 4 longitudinal steel bars of diameter 20 mm. It carries an axial compressive load $P = 600$ kN. Modular ratio $m = E_s/E_c = 15$. Determine the stresses in steel (σ_s) and concrete (σ_c), and load percentage carried by steel.

Solution Breakdown

1. Gross Area $A = 300 \times 300 = 90,000 \text{ mm}^2$.
2. Steel Area $A_s = 4 \times \frac{\pi}{4}(20)^2 = 1256.64 \text{ mm}^2$.
3. Net Concrete Area $A_c = A - A_s = 90,000 - 1256.64 = 88,743.36 \text{ mm}^2$.
4. Stress Ratio: $\sigma_s = 15 \cdot \sigma_c$.
5. Equilibrium:

$$P = \sigma_s A_s + \sigma_c A_c = 15\sigma_c(1256.64) + \sigma_c(88,743.36) = 107,592.96 \cdot \sigma_c$$

$$600 \times 10^3 \text{ N} = 107,592.96 \cdot \sigma_c \implies \sigma_c = 5.58 \text{ MPa}$$

$$\sigma_s = 15 \times 5.58 = 83.70 \text{ MPa}$$

6. Load Share: $P_s = 83.70 \times 1256.64 = 105.18 \text{ kN } (17.53\%)$, $P_c = 494.82 \text{ kN } (82.47\%)$.

- **Core Concepts Summarized:**
  - Composite sections rely on *rigid bonding* and *strain compatibility* ($\varepsilon_1 = \varepsilon_2$).
  - Stress divides according to stiffness: $\sigma_1 = m \cdot \sigma_2$, where $m = E_1/E_2$.
  - Total load $P = \sigma_2 A_{eq} = \sigma_2(mA_1 + A_2)$.
- **Key Formulas Quick Reference:**
  - Modular ratio: $m = \frac{E_1}{E_2}$.
  - Stress compatibility: $\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \implies \sigma_1 = m\sigma_2$.
  - Transformed equivalent area: $A_{eq} = mA_1 + A_2$.
  - Deformation: $\delta L = \frac{PL}{E_1A_1+E_2A_2} = \frac{\sigma_1 L}{E_1} = \frac{\sigma_2 L}{E_2}$.
- **Practical Engineering Insight:**
  - High-E material (like structural steel) acts as a high-capacity stress carrier, drastically increasing load-bearing capacity without proportional increase in gross cross-sectional dimensions.

- **Physical Mechanism of Thermal Expansion**:
  - When a solid material experiences a temperature increase ($\Delta T > 0$), its constituent atoms gain thermal energy and vibrate with greater amplitude.
  - This increased atomic motion causes an increase in average interatomic spacing, manifesting as macroscopic dimensional expansion.
  - Conversely, a reduction in temperature ($\Delta T < 0$) leads to volumetric and linear thermal contraction.
- **Linear Thermal Expansion Equation**:
  - For a homogeneous, isotropic bar of initial length L, the free thermal elongation δ_T is directly proportional to length and temperature change:

$$\delta_T = L \cdot \alpha \cdot \Delta T$$

  - Where α is the **Coefficient of Linear Thermal Expansion** (expressed in K^{-1} or $^{\circ}\mathrm{C}^{-1}$), and $\Delta T = T_{\text{final}} - T_{\text{initial}}$.
- **Concept of Thermal Strain**:

- The unconstrained thermal strain ε_T is defined as the change in length per unit length:

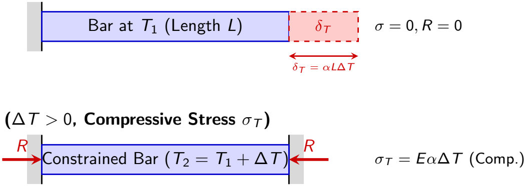
$$\varepsilon_T = \frac{\delta_T}{L} = \alpha \cdot \Delta T$$

- **Fundamental Axiom of Thermal Stress**:
  - *Free expansion induces ZERO thermal stress.* If a body expands or contracts freely without external constraint, $\sigma_T = 0$ despite the presence of thermal strain ε_T.
  - Thermal stresses develop **only** when free thermal deformation is partially or completely restrained by external supports or adjoining structural elements.

- **Principle of Superposition for Thermal Constraint**:
    - To analyze a constrained member, the process can be conceptualized in two distinct hypothetical steps:
        1. **Step 1 (Free Expansion)**: Allow the bar to expand freely by $\delta_T = \alpha L \Delta T$ under temperature change ΔT.
        2. **Step 2 (Mechanical Restraint)**: Apply an external compressive force R to push the expanded end back to its fixed boundary position.
- **Visual Representation of Expansion States**:

(a) Free Thermal Expansion ($\Delta T > 0$, Stress $\sigma_T = 0$)

(b) Rigidly Constrained ($\Delta T > 0$, Compressive Stress σ_T)



- **Mathematical Derivation of Thermal Stress**:
  - Consider a bar of cross-sectional area A, length L, Young's modulus E, and expansion coefficient α, held between two unyielding rigid walls.
  - Free elongation due to heating:
    $$\delta_T = L \cdot \alpha \cdot \Delta T$$
  - Mechanical contraction produced by rigid support reaction R:
    $$\delta_M = \frac{R \cdot L}{A \cdot E} = \frac{\sigma_T \cdot L}{E}$$
  - **Kinematic Compatibility Condition**: Net change in length between rigid supports must equal zero:
    $$\delta_{\text{net}} = \delta_T - \delta_M = 0 \implies L \cdot \alpha \cdot \Delta T - \frac{\sigma_T \cdot L}{E} = 0$$
- **Governing Thermal Stress Equation**:
  $$\sigma_T = E \cdot \alpha \cdot \Delta T$$
- **Support Reaction Force**:

$$R = A \cdot \sigma_T = A \cdot E \cdot \alpha \cdot \Delta T$$

- **Crucial Characteristics of Non-Yielding Thermal Stress**:
  - **Length Independence**: The magnitude of thermal stress σ_T is completely independent of the bar length L!
  - **Sign Convention**: Temperature rise ($\Delta T > 0$) induces **compressive stress**, while temperature drop ($\Delta T < 0$) induces **tensile stress**.

- **Material Selection Impact on Thermal Stress**:
  - The thermal stress generated in a rigid constraint is governed by the product $E \cdot \alpha$.
  - Materials with high Young's modulus and high thermal expansion coefficients develop extreme internal stresses under moderate temperature fluctuations.

Table: Engineering Properties and Induced Thermal Stress for $\Delta T = 50°\text{C}$ (Rigid Support)

Engineering Material	Elastic Modulus E (GPa)	Expansion Coeff. α ($10^{-6}/°\text{C}$)	Product $E \cdot \alpha$ (MPa/°C)	Thermal Stress σ_T (MPa)
Structural Steel (E250)	200	12.0	2.40	120.0 (Compressive)
Aluminum Alloy (6061-T6)	70	23.0	1.61	80.5 (Compressive)
Copper (C11000)	110	17.0	1.87	93.5 (Compressive)
Brass (C36000)	100	19.0	1.90	95.0 (Compressive)
Gray Cast Iron (FG 200)	110	10.5	1.16	57.8 (Compressive)
Stainless Steel (AISI 304)	193	17.3	3.34	167.0 (Compressive)
Titanium Alloy (Ti-6Al-4V)	114	8.6	0.98	49.0 (Compressive)

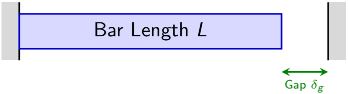
- **Engineering Insights**:

- Stainless Steel develops the highest thermal stress (167 MPa) due to its combination of high stiffness and thermal expansion rate.
- Structural Steel experiences 120 MPa for a 50°C shift—approaching half of its yield strength (250 MPa).

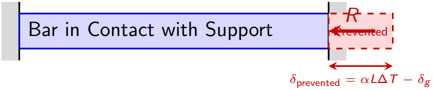
- **Concept of Partial Restraint and Support Yielding**:
    - In real structures, supports are rarely perfectly rigid; they may deform elastically/plastically by a distance δ_y.
    - Alternatively, engineers deliberately provide an expansion gap δ_g (e.g., in railway tracks or piping systems) to absorb free thermal expansion.
- **Kinematic Compatibility for Yielding Support / Gap Allowance**:
    - Free thermal expansion attempted by the bar: $\delta_T = L \cdot \alpha \cdot \Delta T$.
    - If $\delta_T \leq \delta_g$, the bar expands freely into the gap without contacting the support $\implies \sigma_T = 0$.
    - If $\delta_T > \delta_g$ (or support yields by δ_y), net restricted deformation is:

$$\delta_{\text{prevented}} = \delta_T - \delta_g = L \cdot \alpha \cdot \Delta T - \delta_g$$

(a) Unheated State (T_1) with Initial Expansion Gap δ_g



(b) Heated State ($T_2 = T_1 + \Delta T$) with Restricted Deformation



- **Thermal Stress Equation for Yielding Support / Gap Allowance**:
  - The restricted strain ε_{net} experienced by the bar is:
  $$\varepsilon_{\text{net}} = \frac{\delta_{\text{prevented}}}{L} = \frac{L \cdot \alpha \cdot \Delta T - \delta_g}{L} = \alpha \cdot \Delta T - \frac{\delta_g}{L}$$
  - Thermal stress developed:
  $$\sigma_T = E \cdot \varepsilon_{\text{net}} = E \left(\alpha \cdot \Delta T - \frac{\delta_g}{L} \right) = \frac{E}{L} (\alpha \cdot L \cdot \Delta T - \delta_g)$$
  - Resulting reaction force at the support:
  $$R = A \cdot \sigma_T = \frac{A \cdot E}{L} (\alpha \cdot L \cdot \Delta T - \delta_g)$$
- **Comparative Summary of Boundary Conditions**:
  - **Non-Yielding (Rigid Walls, $\delta_g = 0$)**:
  $$\sigma_T = E \cdot \alpha \cdot \Delta T$$
  - **Yielding Support / Allowance Gap ($\delta_g > 0$)**:
  $$\sigma_T = \frac{E}{L} (\alpha \cdot L \cdot \Delta T - \delta_g)$$
  - **Free Unconstrained Expansion ($\delta_g \geq \alpha L \Delta T$)**:
  $$\sigma_T = 0, \quad R = 0$$

- **Problem Statement**:
  - A structural steel rod of length $L = 2.0$ m (2000 mm) and uniform cross-sectional area $A = 400$ mm^2 is rigidly fixed between two unyielding supports at an initial temperature of 25°C.
  - The temperature of the rod is subsequently raised to 95°C.
  - **Material Properties**: Young's modulus $E = 200$ GPa $= 2 \times 10^5$ N/mm^2, Coefficient of linear expansion $\alpha = 12 \times 10^{-6}/°$C.
  - **Determine**:
    1. Free thermal expansion of the rod (δ_T).
    2. Thermal stress induced in the rod (σ_T).
    3. Magnitude and nature of the support reaction force (R).
- **Step-by-Step Solution**:
  - **Step 1: Calculate Temperature Change**:

$$\Delta T = T_{\text{final}} - T_{\text{initial}} = 95°\text{C} - 25°\text{C} = 70°\text{C}$$

  - **Step 2: Calculate Free Expansion (δ_T)**:

$$\delta_T = L \cdot \alpha \cdot \Delta T = 2000 \text{ mm} \times (12 \times 10^{-6}/^\circ\text{C}) \times 70^\circ\text{C} = 1.68 \text{ mm}$$

- **Step 3: Calculate Thermal Stress (σ_T)**:

$$\sigma_T = E \cdot \alpha \cdot \Delta T = (200 \times 10^3 \text{ MPa}) \times (12 \times 10^{-6}) \times 70 = 168.0 \text{ MPa}$$

  Since temperature increases, the stress is **Compressive**.

- **Step 4: Calculate Reaction Force (R)**:

$$R = \sigma_T \cdot A = 168.0 \text{ N/mm}^2 \times 400 \text{ mm}^2 = 67200 \text{ N} = 67.20 \text{ kN}$$

- **Problem Statement**:
  - A heavy copper bar of length $L = 1.5$ m (1500 mm) and diameter $d = 30$ mm is rigidly anchored at one end. A gap of $\delta_g = 0.75$ mm exists between the free end and a rigid stop at 20°C.
  - The temperature of the copper bar increases to 100°C.
  - **Material Data**: $E = 110$ GPa $= 1.1 \times 10^5$ N/mm^2, $\alpha = 17 \times 10^{-6}/°$C.
  - **Calculate**:
    1. Free thermal expansion (δ_T) and check for support contact.
    2. Net thermal stress induced in the copper bar (σ_T).
    3. Total compressive force exerted on the support (R).
- **Step-by-Step Solution**:
  - **Step 1: Temperature Rise & Free Expansion**:

$$\Delta T = 100°\text{C} - 20°\text{C} = 80°\text{C}$$

$$\delta_T = L \cdot \alpha \cdot \Delta T = 1500 \text{ mm} \times (17 \times 10^{-6}) \times 80 = 2.04 \text{ mm}$$

Since $\delta_T = 2.04$ mm $> \delta_g = 0.75$ mm, contact occurs and thermal stress is generated!

- **Step 2: Restricted Expansion ($\delta_{\text{prevented}}$)**:

$$\delta_{\text{prevented}} = \delta_T - \delta_g = 2.04 \text{ mm} - 0.75 \text{ mm} = 1.29 \text{ mm}$$

- **Step 3: Thermal Stress Calculation (σ_T)**:

$$\sigma_T = \frac{E}{L} \cdot \delta_{\text{prevented}} = \frac{110 \times 10^3 \text{ MPa}}{1500 \text{ mm}} \times 1.29 \text{ mm} = 94.60 \text{ MPa} \quad \text{(Compressive)}$$

- **Step 4: Support Reaction Force (R)**:

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (30 \text{ mm})^2 = 706.86 \text{ mm}^2$$

$$R = \sigma_T \cdot A = 94.60 \text{ N/mm}^2 \times 706.86 \text{ mm}^2 = 66869 \text{ N} = 66.87 \text{ kN}$$

- **Real-World Engineering Challenges**:
  - **Continuous Welded Rails (CWR)**: Long rail sections experience severe compressive thermal stress in hot summers, leading to track buckling ("sun kinks") if ballast resistance is inadequate.
  - **High-Temperature Piping Systems**: Steam pipelines in thermal power plants undergo substantial thermal growth; rigid anchoring would cause structural fracture.
  - **Long-Span Bridges**: Concrete and steel bridge decks expand and contract daily and seasonally.
- **Thermal Stress Mitigation Strategies**:
  - **Expansion Joints & Slotted Holes**: Installing comb-type expansion joints and roller bearings on bridges to allow free longitudinal displacement ($\sigma_T = 0$).
  - **Expansion Loops & Bellows**: Incorporating flexible U-shaped expansion loops or corrugated metallic bellows into steam piping to absorb thermal elongation via flexure.

- **Material Selection & Matching**: Matching thermal expansion coefficients ($\alpha_1 \approx \alpha_2$) in bimetallic strip applications or composite matrix interfaces to minimize interfacial shear stresses.

Course Context: GTU DI03000201 – Direct Stress and Strain

Unit 1: Direct Stress and Strain explores the structural response of machine and civil components subjected to different types of applied loads: gradual (static), sudden (dynamic), and impact (shock) loads.

Key Lecture Topics

- **Load Classifications**: Mathematical definitions and stress-strain behavior under gradual, sudden, and impact loads.
- **Deformation Analysis**: Quantifying instantaneous extension $(\delta_g, \delta_s, \delta_i)$ and impact load factor (i_f).
- **Strain Energy (U)**: Internal energy stored in elastic bodies due to mechanical work done by external forces.
- **Resilience Properties**: Resilience, Proof Resilience (U_p), and Modulus of Resilience (u_r).
- **GTU Solved Numerical Problems**: Step-by-step solutions for impact energy absorption and design threshold calculations.

Primary Learning Objectives

Derive dynamic load formulas, evaluate impact stress multipliers, analyze material resilience performance using real engineering data, and calculate safety margins against shock loading.

1. Gradual Loading (Static Load)

Load is applied in infinitesimal increments from 0 to P. Average load acting during deformation is $\frac{P}{2}$.

- **Work Done (W)**: $W = \frac{1}{2} P \cdot \delta_g$
- **Strain Energy (U)**: $U = \frac{\sigma_g^2}{2E} \cdot V$, where $V = A \cdot L$
- **Equating Work and Energy**:

$$\frac{1}{2} P \cdot \delta_g = \frac{\sigma_g^2}{2E}(AL) \implies \sigma_g = \frac{P}{A}, \quad \delta_g = \frac{PL}{AE}$$

2. Sudden Loading (Instantaneous Dynamic Load)

Full load magnitude P is applied instantly to the member without any drop height ($h = 0$).

- **Work Done (W)**: $W = P \cdot \delta_s$
- **Strain Energy (U)**: $U = \frac{\sigma_s^2}{2E} \cdot V = \frac{1}{2} P \delta_s$
- **Derivation of Sudden Stress (σ_s)**:

$$P \cdot \delta_s = \frac{\sigma_s^2}{2E}(AL) = \frac{1}{2}\sigma_s A \cdot \left(\frac{\sigma_s L}{E}\right) = \frac{1}{2}\sigma_s A \cdot \delta_s$$

$$\implies \sigma_s = \frac{2P}{A} = 2\sigma_g, \quad \delta_s = \frac{2PL}{AE} = 2\delta_g$$

Critical Engineering Takeaway

Instantaneous stress and deformation under sudden load application are **EXACTLY TWICE** the corresponding static values for the same load magnitude!

Impact Loading Principle

A body of weight $P = mg$ falls through a clear height h onto a collar or flange fixed at the lower end of a vertical rod of cross-sectional area A and length L.

- Total vertical drop distance of falling load $= (h + \delta_i)$
- Potential energy lost by falling weight $= P(h + \delta_i)$

Derivation of Impact Stress (σ_i)

Equating potential energy loss to internal strain energy stored:

$$P(h + \delta_i) = \frac{\sigma_i^2}{2E} \cdot AL$$

Substituting instantaneous extension $\delta_i = \frac{\sigma_i L}{E}$:

$$P\left(h + \frac{\sigma_i L}{E}\right) = \frac{\sigma_i^2 AL}{2E} \implies \left(\frac{AL}{2E}\right)\sigma_i^2 - \left(\frac{PL}{E}\right)\sigma_i - Ph = 0$$

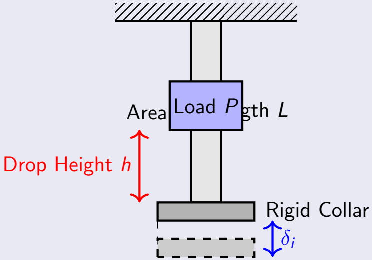
Solving this quadratic equation in σ_i yields:

$$\sigma_i = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{PL}}\right) = \sigma_g\left(1 + \sqrt{1 + \frac{2h}{\delta_g}}\right)$$

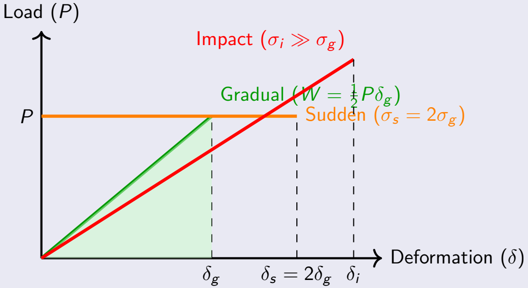
Impact Factor & Deformation

- **Impact Deformation (δ_i)**: $\delta_i = \delta_g \left(1 + \sqrt{1 + \frac{2h}{\delta_g}}\right)$
- **Impact Factor (i_f)**: $i_f = 1 + \sqrt{1 + \frac{2h}{\delta_g}}$ where $\delta_g = \frac{PL}{AE}$
- *Limiting Case 1 ($h = 0$)*: $i_f = 2 \implies \sigma_i = 2\sigma_g$ (Sudden loading).
- *Limiting Case 2 ($h \gg \delta_g$)*: $\sigma_i \approx \sqrt{\frac{2E \cdot Ph}{AL}} = \sqrt{\frac{2E \cdot W_{\text{ext}}}{V}}$.

Impact Loading Collar Setup



Load-Deformation Comparison (P–δ Profiles)



1. Strain Energy (U)

Strain energy is the internal work done by stresses in deforming an elastic body under external load.

$$U = \frac{\sigma^2}{2E} \cdot V = \frac{1}{2} P \cdot \delta \quad \text{(for axial gradual load)}$$

- **Unit**: Joules (J), $\mathrm{N \cdot m}$, or $\mathrm{N \cdot mm}$.
- **Volume Dependence**: Strain energy is directly proportional to total component volume $V = A \cdot L$.

2. Resilience

The capacity of a material to absorb energy under deformation within its elastic limit and release that energy during unloading without permanent set.

3. Proof Resilience (U_p)

The maximum strain energy that a given body can store up to its elastic limit or yield point ($\sigma = \sigma_y$).

$$U_p = \frac{\sigma_y^2}{2E} \cdot V$$

4. Modulus of Resilience (u_r)

The maximum strain energy stored per unit volume at the elastic limit of a material.

$$u_r = \frac{U_p}{V} = \frac{\sigma_y^2}{2E}$$

- **Unit**: J/m^3, N/mm^2, or MJ/m^3.
- Represents an intrinsic material property independent of structural dimensions.

Real Material Properties for Energy Storage & Shock Resistance

Material Specification	σ_y **(MPa)**	E **(GPa)**	ρ **(kg/m^3)**	u_r **(kJ/m^3)**	**Specific** u_r/ρ **(J/kg)**
Structural Steel (Fe 250)	250	200	7850	156.25	19.90
High-Strength Steel (Fe 415)	415	205	7850	419.94	53.50
Spring Steel (55Si7)	1200	206	7850	3495.15	445.24
Aluminum Alloy (6061-T6)	276	68.9	2700	552.76	204.73
Titanium Alloy (Ti-6Al-4V)	880	114	4430	3394.74	766.31
Phosphor Bronze (C51000)	380	110	8800	656.36	74.59

Engineering Design Insights

- **Spring Applications**: Spring Steel (55Si7) yields a Modulus of Resilience $u_r \approx 3.5\ \mathrm{MJ/m^3}$, over 22 times higher than mild steel.
- **Aerospace Dynamic Components**: Titanium (Ti-6Al-4V) provides high specific resilience (766.3 J/kg), offering superior impact energy absorption per unit mass.
- **Modulus Influence**: Lower Young's Modulus (E) combined with high yield strength (σ_y) maximizes energy storage capacity.

GTU Problem Statement

A steel rod of diameter $d = 20$ mm and length $L = 2$ m is subjected to an axial load $P = 15$ kN. Take $E = 200$ GPa $= 2 \times 10^5$ N/mm^2. Calculate instantaneous stress, deformation, and strain energy for:

a. Gradual load application.
b. Sudden load application.
c. Impact load falling from height $h = 50$ mm.

Step 1: Geometrical & Material Parameters

- Area $A = \frac{\pi}{4}(20)^2 = 314.16$ mm^2, Length $L = 2000$ mm, Load $P = 15000$ N.
- Volume $V = A \times L = 314.16 \times 2000 = 628,320$ mm^3.

Step 2: Case (a) Gradual Loading

$$\sigma_g = \frac{P}{A} = \frac{15000}{314.16} = 47.75 \text{ N/mm}^2 \text{ (MPa)}$$

$$\delta_g = \frac{\sigma_g L}{E} = \frac{47.75 \times 2000}{200000} = 0.4775 \text{ mm}$$

$$U_g = \frac{\sigma_g^2}{2E} \cdot V = \frac{(47.75)^2}{2 \times 200000} \times 628320 = 3581.25 \text{ N} \cdot \text{mm} = 3.581 \text{ J}$$

Step 3: Case (b) Sudden Loading

$$\sigma_s = 2\sigma_g = 2 \times 47.75 = 95.50 \text{ MPa}$$

$$\delta_s = 2\delta_g = 2 \times 0.4775 = 0.9550 \text{ mm}$$

$$U_s = \frac{\sigma_s^2}{2E} \cdot V = \frac{(95.50)^2}{2 \times 200000} \times 628320 = 14,325 \text{ N} \cdot \text{mm} = 14.325 \text{ J}$$

Step 4: Case (c) Impact Loading ($h = 50$ mm)

$$\text{Impact Factor } i_f = 1 + \sqrt{1 + \frac{2h}{\delta_g}} = 1 + \sqrt{1 + \frac{2 \times 50}{0.4775}} = 1 + \sqrt{210.42} = 15.506$$

$$\sigma_i = \sigma_g \times i_f = 47.75 \times 15.506 = 740.41 \text{ MPa}$$

$$\delta_i = \delta_g \times i_f = 0.4775 \times 15.506 = 7.404 \text{ mm}$$

$$U_i = \frac{\sigma_i^2}{2E} \cdot V = \frac{(740.41)^2}{2 \times 200000} \times 628320 = 861,130 \text{ N} \cdot \text{mm} = 861.13 \text{ J}$$

GTU Problem Statement

An alloy steel bar of uniform diameter $d = 30$ mm and length $L = 1.5$ m has an elastic yield limit $\sigma_y = 300$ MPa and Young's Modulus $E = 210$ GPa $= 2.1 \times 10^5$ N/mm^2.

- (a) Calculate Proof Resilience (U_p) and Modulus of Resilience (u_r).
- (b) Determine maximum drop height $h_{\max}$ for a falling load $P = 5$ kN onto the bottom collar without exceeding the elastic limit.

Step 1: Evaluation of Resilience Properties

- Area $A = \frac{\pi}{4}(30)^2 = 706.86 \text{ mm}^2$, Length $L = 1500$ mm.
- Volume $V = 706.86 \times 1500 = 1,060,290 \text{ mm}^3$.
- **Modulus of Resilience**:

$$u_r = \frac{\sigma_y^2}{2E} = \frac{300^2}{2 \times 210000} = 0.2143 \text{ N/mm}^2 = 214.3 \text{ kJ/m}^3$$

- **Proof Resilience**:

$$U_p = u_r \times V = 0.2143 \times 1060290 = 227,220 \text{ N} \cdot \text{mm} = 227.22 \text{ J}$$

Step 2: Maximum Permissible Drop Height h_{max}

To ensure no permanent deformation occurs, maximum impact stress $\sigma_i = \sigma_y = 300$ MPa.

- Elastic extension at yield limit:

$$\delta_i = \frac{\sigma_y L}{E} = \frac{300 \times 1500}{210000} = 2.143 \text{ mm}$$

- Work done by falling load $P = 5000$ N:

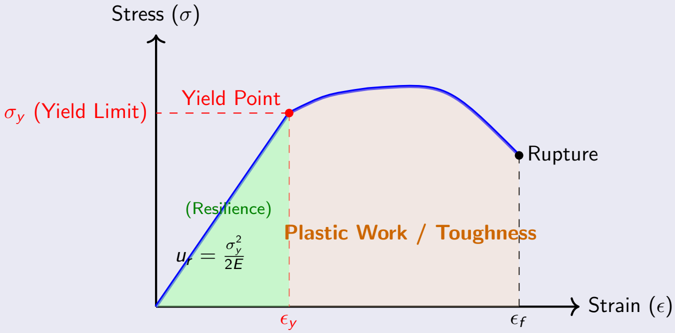
$$W_{\text{ext}} = P(h_{\text{max}} + \delta_i) = 5000(h_{\text{max}} + 2.143) \text{ N} \cdot \text{mm}$$

- Equating work done to Proof Resilience U_p:

$$5000(h_{\text{max}} + 2.143) = 227220 \implies h_{\text{max}} + 2.143 = 45.444 \text{ mm}$$

$$h_{\text{max}} = 45.444 - 2.143 = 43.301 \text{ mm} \quad (\approx 4.33 \text{ cm})$$

Graphical Representation of Resilience & Toughness



Comparison of Energy Storage Metrics

- **Modulus of Resilience (u_r)**: Triangular elastic area under σ–ϵ curve up to yield point σ_y. Fully recoverable upon unloading.
- **Modulus of Toughness**: Total area under σ–ϵ curve up to fracture point ϵ_f. Includes elastic strain energy plus plastic dissipation energy prior to fracture.
- **Impact Loading Limit**: Mechanical structures designed for cyclic dynamic loads must operate within u_r bounds to prevent fatigue damage and permanent set.

What is Moment of Inertia?

- **Second Moment of Area / Mass:** A fundamental geometric and dynamic property representing how area or mass is distributed relative to a given axis.
- **Physical Intuition:** Measure of resistance of a body to rotational acceleration or structural flexural bending.
- **Mathematical Concept:** Derived by taking the second moment of elemental units ($\int r^2\, dA$ or $\int r^2\, dm$).

Dual Concepts in Engineering Mechanics

- **Area Moment of Inertia (I or J):** Purely geometric cross-sectional property governing bending resistance of beams and torsional resistance of shafts. Units: $\mathrm{m}^4, \mathrm{mm}^4, \mathrm{cm}^4$.
- **Mass Moment of Inertia (I_{mass}):** Dynamic property governing angular acceleration of rotating rigid bodies (flywheels, rotors, gears). Units: $\mathrm{kg} \cdot \mathrm{m}^2$.

Linear vs. Angular Analogy

- **Linear Translation:** Force $F = m \cdot a$, where mass m resists linear acceleration.
- **Rotational Motion:** Torque $\tau = I \cdot \alpha$, where moment of inertia I resists angular acceleration.
- **Structural Bending:** Bending Moment $M = \frac{E \cdot I}{\rho}$, where I resists curvature and flexural deformation.

Second Moment of Area Equations

For a planar cross-sectional area A located in the x-y Cartesian plane:

- **Moment of Inertia about x-axis (I_x):**
$$I_x = \int_A y^2 \, dA$$
- **Moment of Inertia about y-axis (I_y):**
$$I_y = \int_A x^2 \, dA$$

Polar Moment of Inertia (J_z or I_p)

- Moment of inertia about an axis perpendicular to the cross-section passing through origin O (z-axis):

$$J_z = \int_A r^2 \, dA = \int_A (x^2 + y^2) \, dA = I_x + I_y$$

- Key parameter in determining torsional shear stress $\tau = \frac{T \cdot r}{J_z}$ and torsional rigidity ($G \cdot J_z$).

Radius of Gyration (k)

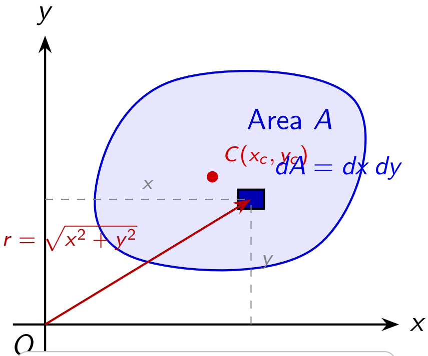
- The imaginary distance k from the reference axis at which the total area A could be concentrated to give the same moment of inertia:

$$k_x = \sqrt{\frac{I_x}{A}}, \quad k_y = \sqrt{\frac{I_y}{A}} \implies I = A \cdot k^2$$

- Essential in structural column design for assessing slenderness ratio ($\lambda = \frac{L_e}{k}$) and buckling stability.

Geometric Definition

Consider an arbitrary 2D cross-section A in the x-y plane with a centroid $C(x_c, y_c)$ and elemental area dA:



$I_x = \int_A y^2\,dA$ (about x-axis)
$I_y = \int_A x^2\,dA$ (about y-axis)
$J_z = \int_A r^2\,dA = I_x + I_y$ (Polar MOI)

Centroidal Moments of Inertia for Common Cross-Sections

Shape	**Area** (A)	I_{xc} **(Centroidal** x**)**	I_{yc} **(Centroidal** y**)**	**Polar** J_c
Rectangle $(b \times h)$	$b \cdot h$	$\frac{bh^3}{12}$	$\frac{hb^3}{12}$	$\frac{bh(b^2+h^2)}{12}$
Circle (Diameter d)	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	$\frac{\pi d^4}{64}$	$\frac{\pi d^4}{32}$
Hollow Circle (D, d)	$\frac{\pi(D^2-d^2)}{4}$	$\frac{\pi(D^4-d^4)}{64}$	$\frac{\pi(D^4-d^4)}{64}$	$\frac{\pi(D^4-d^4)}{32}$
Triangle (Base b, Height h)	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{hb^3}{48}$	$\frac{bh^3}{36} + \frac{hb^3}{48}$
Semicircle (Radius R)	$\frac{\pi R^2}{2}$	$\left(\frac{9\pi^2-64}{72\pi}\right) R^4 \approx 0.1098R^4$	$\frac{\pi R^4}{8}$	$I_{xc} + I_{yc}$

1. Parallel Axis Theorem (Steiner's Theorem)

- Relates the moment of inertia about any arbitrary axis to the moment of inertia about a parallel axis passing through the centroid:

$$I_{\text{axis}} = I_c + A \cdot d^2$$

  where I_c is the centroidal moment of inertia, A is the cross-sectional area, and d is the perpendicular distance between the parallel axes.

- **Mandatory Rule:** The baseline term I_c MUST be about the centroidal axis of the section.

2. Perpendicular Axis Theorem

- For a 2D planar lamina lying in the *x*-*y* plane, the moment of inertia about the mutually perpendicular *z*-axis is:

$$I_z = I_x + I_y$$

- **Applicability Limit:** Valid ONLY for thin planar (2D) areas, NOT for 3D solid structures.

Common Engineering Misconceptions

- *Incorrect:* Applying $I = I_{\text{base}} + Ad^2$ where I_{base} is not centroidal.
- *Correct Method:* Always transfer to centroid first: $I_{\text{new}} = (I_{\text{given}} - Ad_1^2) + Ad_2^2$.

Comparative Analysis

Parameter	**Area Moment of Inertia (I_{area})**	**Mass Moment of Inertia (I_{mass})**
Formula	$I = \int r^2 \, dA$	$I = \int r^2 \, dm$
SI Units	m^4 or mm^4	$\text{kg} \cdot \text{m}^2$
Physical Basis	Cross-sectional geometry distribution	Mass density distribution in 3D space
Engineering Field	Civil / Structural	Mechanical / Aerospace Dynamics
Primary Application	Beam bending stress ($\sigma = \frac{My}{I}$)	Dynamic torque ($\tau = I\alpha$) & Kinetic energy ($E_k = \frac{1}{2} I \omega^2$)

Flywheel Energy Storage Principle

- Kinetic energy stored in a spinning flywheel: $E_k = \frac{1}{2} I_{\text{mass}} \cdot \omega^2$.
- To maximize energy storage without adding unnecessary weight, mass is concentrated at outer rim radius R, yielding $I \approx m \cdot R^2$.

1. Flexural Bending Resistance in Beams

- Euler-Bernoulli flexural stress formula: $\sigma = \frac{M \cdot y}{I_x}$.
- Maximum bending stress $\sigma_{\text{max}} = \frac{M}{Z}$, where Section Modulus $Z = \frac{I_x}{y_{\text{max}}}$.
- High I_x minimizes internal tensile and compressive stresses under external loading.

2. Torsional Resistance in Drive Shafts

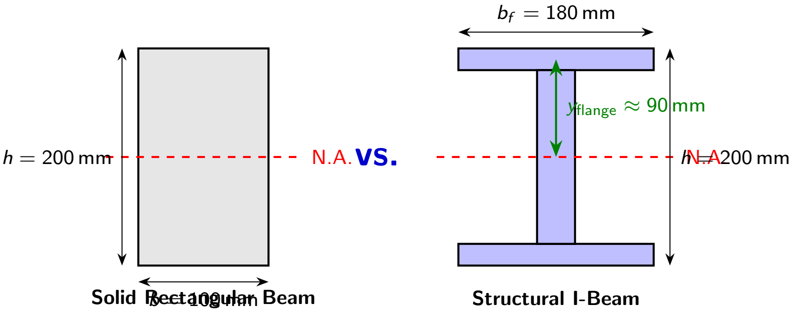
- Torsional shear stress formula: $\tau = \frac{T \cdot r}{J_z}$.
- Hollow circular shafts provide higher polar moment J_z per unit mass than solid shafts, optimizing weight-to-strength ratio in drive shafts.

3. Structural Column Buckling

- Euler critical buckling load: $P_{cr} = \frac{\pi^2 E I_{min}}{L_e^2}$.
- Columns always buckle about the axis with the **minimum** moment of inertia (I_{min}).

Why I-Beams Dominates Structural Engineering

Material located far from the neutral axis (N.A.) contributes quadratically (y^2) to $I_x = \int y^2 \, dA$, maximizing stiffness per unit material weight.



Solid Rectangular Beam

Area $A = 20,000\,\text{mm}^2$

$I_x = \frac{bh^3}{12} = 6.67 \times 10^7\,\text{mm}^4$

Material near N.A. underutilized

Structural I-Beam

Area $A = 20,000\,\text{mm}^2$ (Equal Weight!)

$I_x \approx 2.45 \times 10^8\,\text{mm}^4$

$\approx 3.7\times$ **Higher Bending Stiffness!**

Properties of Standard Indian Structural Steel Profiles (IS 808)

Section Designation	Mass (kg/m)	Area (cm^2)	I_x (cm^4)	I_y (cm^4)	r_x (cm)	r_y (cm)
ISMB 100 (100 × 75 mm)	11.5	14.6	257.5	40.8	4.20	1.67
ISMB 200 (200 × 100 mm)	25.4	32.3	2235.4	150.0	8.32	2.15
ISMB 300 (300 × 140 mm)	44.2	56.3	8603.6	453.9	12.37	2.84
ISMC 150 (Channel section)	16.4	20.8	778.3	102.3	6.12	2.22
Steel Pipe (114.3 OD × 4.5 t)	12.2	15.5	236.7	236.7	3.91	3.91

Design Observations from Structural Data

- **Anisotropy of Stiffness:** For ISMB 300, $I_x/I_y \approx 19.0$. I-beams are exceptionally strong against major-axis bending (x-x), but require lateral bracing against minor-axis (y-y) buckling.
- **Pipe Symmetry:** Circular steel pipes feature identical $I_x = I_y$ ($r_x = r_y$), making them ideal for multi-directional compression columns.

Course Information

- **Course Code**: DI03000201 – Engineering Mechanics / Strength of Materials
- **Unit**: Moment of Inertia
- **Topic**: Concept of Section Modulus (Z), Radius of Gyration (k), and Polar Moment of Inertia (J)

Learning Objectives

- Formulate the mathematical definition and physical significance of **Section Modulus** (Z) in bending.
- Derive Section Modulus formulas for standard geometric shapes (rectangular, circular, hollow, and I-sections).
- Explain the concept of **Radius of Gyration** (k) and its essential role in column buckling analysis.
- Define **Polar Moment of Inertia** (J) and Polar Section Modulus (Z_p) for torsional shear stress calculation.
- Evaluate and compare real engineering section properties for structural optimization.

Definition & Mathematical Expression

The **Section Modulus** (Z) is the ratio of the second moment of area (Moment of Inertia, I) about the neutral axis to the distance (y_{max}) from the neutral axis to the outermost fiber of the cross-section:

$$Z = \frac{I}{y_{\text{max}}}$$

Units of Section Modulus: mm^3, cm^3, or m^3.

Relationship with Flexure (Bending) Formula

From Euler-Bernoulli beam theory:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \implies M = \sigma_{\text{max}} \cdot \left(\frac{I}{y_{\text{max}}}\right) = \sigma_{\text{max}} \cdot Z$$

where:

- M = Bending moment resisting capacity ($\text{N} \cdot \text{mm}$)
- σ_{max} = Maximum allowable bending stress (N/mm^2 or MPa)
- Z = Section Modulus (mm^3)

Engineering Significance

- Z represents the **bending strength efficiency** of a beam cross-section.
- Higher Z value indicates greater moment-carrying capacity for a given material stress.
- Unsymmetrical sections have two moduli: $Z_{top} = I/y_{top}$ and $Z_{bottom} = I/y_{bottom}$. Bending strength is limited by the smaller section modulus.

Table: Section Modulus Formulas for Standard Geometric Sections

Cross-Section Shape	**Moment of Inertia (I_{xx})**	**Extreme Fiber ($y_{\max}$)**	**Section Modulus (Z_{xx})**
Solid Rectangle ($b \times h$)	$\frac{bh^3}{12}$	$\frac{h}{2}$	$\frac{bh^2}{6}$
Solid Circle (Diameter d)	$\frac{\pi d^4}{64}$	$\frac{d}{2}$	$\frac{\pi d^3}{32}$
Hollow Circle (D, d)	$\frac{\pi(D^4 - d^4)}{64}$	$\frac{D}{2}$	$\frac{\pi(D^4 - d^4)}{32D}$
Hollow Rectangle (B, H, b, h)	$\frac{BH^3 - bh^3}{12}$	$\frac{H}{2}$	$\frac{BH^3 - bh^3}{6H}$
Triangle (Base b, Height h)	$\frac{bh^3}{36}$ (at NA)	$\frac{2h}{3}$ (to apex)	$\frac{bh^2}{24}$

Key Design Insight

For a fixed cross-sectional area, placing material as far as possible from the neutral axis (increasing y) maximizes I faster than $y_{\max}$, resulting in a significantly higher Section Modulus Z. This makes I-beams and hollow tubes highly efficient in bending.

Definition & Physical Interpretation

The **Radius of Gyration** (k or r) is the distance from a reference axis at which the entire area A of a cross-section could be concentrated as a point mass/area without changing its moment of inertia I about that axis:

$$I = A \cdot k^2 \implies k = \sqrt{\frac{I}{A}}$$

Units of Radius of Gyration: mm, cm, or m.

Principal Axes Radii of Gyration

- About Centroidal x-axis: $k_{xx} = \sqrt{\frac{I_{xx}}{A}}$
- About Centroidal y-axis: $k_{yy} = \sqrt{\frac{I_{yy}}{A}}$
- Polar Radius of Gyration: $k_z = \sqrt{\frac{J}{A}} = \sqrt{k_{xx}^2 + k_{yy}^2}$

Application in Column Buckling Theory

In structural engineering, column failure under axial compression depends on the **Slenderness Ratio** (λ):

$$\lambda = \frac{L_{\text{eff}}}{k_{\text{min}}} \quad \text{where } k_{\text{min}} = \min(k_{xx}, k_{yy})$$

Euler's critical buckling load: $P_{\text{cr}} = \frac{\pi^2 EI_{\text{min}}}{L_{\text{eff}}^2} = \frac{\pi^2 EA}{\lambda^2}$. Columns always buckle about the axis with the **minimum radius of gyration** (k_{min}).

Definition of Polar Moment of Inertia (J)

The **Polar Moment of Inertia** (J or I_z) measures a cross-section's resistance to **torsional deformation** and rotational deflection about an axis perpendicular to its plane (z-axis passing through centroid O):

$$J = I_{zz} = \iint r^2 \, dA$$

Perpendicular Axis Theorem Relationship

For any plane surface lying in the xy-plane:

$$J = I_{zz} = I_{xx} + I_{yy}$$

Torsion Formula & Polar Section Modulus (Z_p)

From the elastic torsion formula:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \implies T = \tau_{\max} \cdot \left(\frac{J}{r_{\max}}\right) = \tau_{\max} \cdot Z_p$$

where $Z_p = \frac{J}{r_{\max}}$ is the **Polar Section Modulus** (Torsional Strength Modulus).

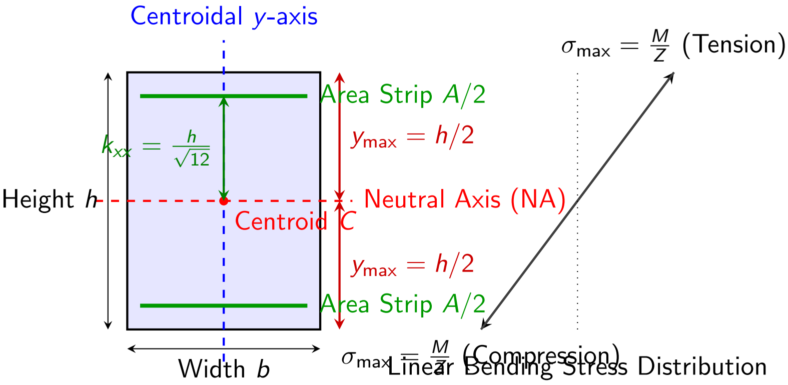
- Solid Shaft (Dia d): $J = \frac{\pi d^4}{32}, \quad Z_p = \frac{\pi d^3}{16}$
- Hollow Shaft (D, d): $J = \frac{\pi(D^4 - d^4)}{32}, \quad Z_p = \frac{\pi(D^4 - d^4)}{16D}$

Table: Real Engineering Properties of Standard Structural Steel Profiles

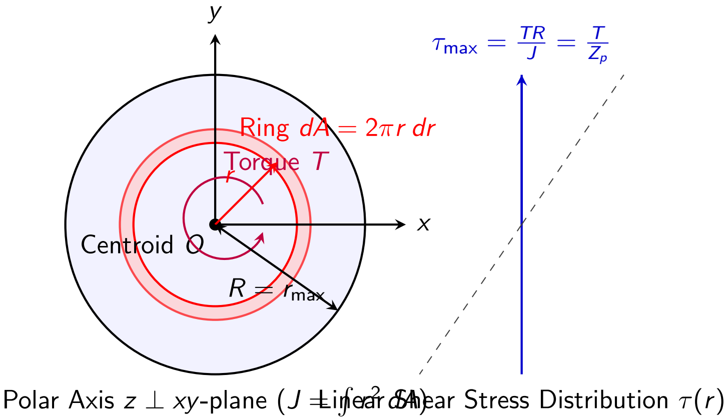
Section Description	A **(cm^2)**	I_{xx} **(cm^4)**	I_{yy} **(cm^4)**	Z_{xx} **(cm^3)**	k_{xx} **(cm)**	k_{yy} **(cm)**	J **(cm^4)**
Solid Round Bar (ϕ100 mm)	78.54	490.87	490.87	98.17	2.50	2.50	981.75
Hollow Tube (100 $\times$ 100 $\times$ 6 mm)	22.56	341.20	341.20	68.24	3.89	3.89	682.40
ISMB 200 (I-Beam)	32.30	2235.40	150.00	223.54	8.32	2.15	2385.40
ISA 100 $\times$ 100 $\times$ 10 mm (Angle)	19.03	177.00	177.00	24.70	3.05	3.05	354.00

Comparative Engineering Analysis

- **Flexural Efficiency**: ISMB 200 provides $Z_{xx} = 223.54$ cm^3 with only $A = 32.30$ cm^2, making it 2.28$\times$ stronger in bending than a solid bar that uses 2.43$\times$ more material!
- **Column Efficiency**: Hollow Tube achieves $k_{\min} = 3.89$ cm compared to 2.50 cm for solid bar, drastically reducing column slenderness λ.
- **Torsional Efficiency**: Circular and square closed tubes maximize polar moment of inertia J per unit mass.



Centroidal y-axis
$\sigma_{\max} = \frac{M}{Z}$ (Tension)
Area Strip $A/2$
$y_{\max} = h/2$
$k_{xx} = \frac{h}{\sqrt{12}}$
Height h
Neutral Axis (NA)
Centroid C
$y_{\max} = h/2$
Area Strip $A/2$
$\sigma_{\max} = \frac{M}{Z}$ (Compression)
Width b
Linear Bending Stress Distribution



y
x
$\tau_{\max} = \frac{TR}{J} = \frac{T}{Z_p}$
Ring $dA = 2\pi r\,dr$
Torque T
r
Centroid O
$R = r_{\max}$
Polar Axis $z \perp xy$-plane ($J = \int r^2\,dA$)
Linear Shear Stress Distribution $\tau(r)$

Problem Statement

A hollow rectangular structural steel column has external dimensions $B = 150$ mm, $H = 250$ mm, and a uniform wall thickness $t = 10$ mm. Determine: (a) Cross-sectional area A, (b) Moments of inertia I_{xx} and I_{yy}, (c) Section moduli Z_{xx} and Z_{yy}, (d) Radii of gyration k_{xx} and k_{yy}, and (e) Polar moment of inertia J.

Solution Steps

1. **Inner Dimensions**:

$$b = B - 2t = 150 - 2(10) = 130 \text{ mm}, \quad h = H - 2t = 250 - 2(10) = 230 \text{ mm}$$

2. **Cross-Sectional Area (A)**:

$$A = B \cdot H - b \cdot h = (150)(250) - (130)(230) = 37,500 - 29,900 = 7,600 \text{ mm}^2 = 76.0 \text{ cm}^2$$

3. **Moments of Inertia (I_{xx}, I_{yy})**:

$$I_{xx} = \frac{BH^3 - bh^3}{12} = \frac{150(250)^3 - 130(230)^3}{12} = \frac{2343.75 \times 10^6 - 1581.71 \times 10^6}{12} = 63.50 \times 10^6 \text{ mm}^4 = 6,350 \text{ cm}^4$$

$$I_{yy} = \frac{HB^3 - hb^3}{12} = \frac{250(150)^3 - 230(130)^3}{12} = \frac{843.75 \times 10^6 - 505.31 \times 10^6}{12} = 28.20 \times 10^6 \text{ mm}^4 = 2,820 \text{ cm}^4$$

Solution Steps Cont.

4. **Section Moduli (Z_{xx}, Z_{yy}):**

$$Z_{xx} = \frac{I_{xx}}{H/2} = \frac{63.50 \times 10^6}{125} = 508.0 \times 10^3 \text{ mm}^3 = 508.0 \text{ cm}^3$$

$$Z_{yy} = \frac{I_{yy}}{B/2} = \frac{28.20 \times 10^6}{75} = 376.0 \times 10^3 \text{ mm}^3 = 376.0 \text{ cm}^3$$

5. **Radii of Gyration (k_{xx}, k_{yy}):**

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{63.50 \times 10^6}{7600}} = \sqrt{8355.26} = 91.41 \text{ mm} = 9.14 \text{ cm}$$

$$k_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{28.20 \times 10^6}{7600}} = \sqrt{3710.53} = 60.91 \text{ mm} = 6.09 \text{ cm}$$

6. **Polar Moment of Inertia (J):**

$$J = I_{xx} + I_{yy} = 63.50 \times 10^6 + 28.20 \times 10^6 = 91.70 \times 10^6 \text{ mm}^4 = 9,170 \text{ cm}^4$$

Table: Summary of Derived Moment of Inertia Parameters

Parameter	Symbol	Formula	Primary Physical Meaning	Key Application
Section Modulus	Z	$\frac{I}{y_{\max}}$	Resistance to flexural bending stress	Beam design ($M = \sigma Z$)
Radius of Gyration	k	$\sqrt{\frac{I}{A}}$	Measure of area dispersion from axis	Column buckling ($\lambda = \frac{L}{k}$)
Polar Moment of Inertia	J	$I_{xx} + I_{yy}$	Resistance to torsional twist	Shaft design ($\frac{T}{J} = \frac{\tau}{r}$)
Polar Section Modulus	Z_p	$\frac{J}{r_{\max}}$	Torsional strength capacity	Shaft sizing ($T = \tau_{\max} Z_p$)

Key Engineering Takeaways

1. **Flexural Optimization**: Maximize Z by moving material away from the neutral axis.
2. **Buckling Prevention**: Minimize slenderness λ by choosing sections with high $k_{\min}$.
3. **Torsional Efficiency**: Circular hollow sections provide maximum Z_p per unit weight due to axisymmetric shear stress distribution.

Course Information

- **Course Code**: DI03000201 – Basics of Applied Mechanics
- **Unit**: Moment of Inertia
- **Topic**: Parallel Axis Theorem & Perpendicular Axis Theorem

Learning Objectives

- Understand and formulate the **Parallel Axis Theorem (Steiner's Theorem)** for plane areas.
- Derive and apply the **Perpendicular Axis Theorem** for thin planar laminas.
- Identify the specific domain, validity constraints, and axis definitions for both theorems.
- Calculate planar (I_x, I_y) and polar (J_z) moments of inertia for standard and composite engineering sections.
- Apply real engineering cross-sectional data to structural design and torsional resistance problems.

Definition of Second Moment of Area

The second moment of area (commonly called Area Moment of Inertia) measures a geometric shape's resistance to bending deflection:

- **Moment of Inertia about X-axis**: $I_x = \int y^2 \, dA$
- **Moment of Inertia about Y-axis**: $I_y = \int x^2 \, dA$
- **Units**: Expressed in mm^4, cm^4, or m^4.

Centroidal Axes System

- The **centroid** $G(\bar{x}, \bar{y})$ represents the geometric center of gravity of an area.
- Axes passing through G are called **centroidal axes** ($x_c - x_c$ and $y_c - y_c$).
- The moment of inertia about a centroidal axis is the **minimum** moment of inertia relative to any parallel axis.
- Transforming moments of inertia between non-centroidal parallel axes requires using the centroidal axis as an intermediate reference.

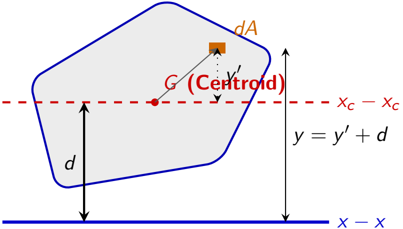
Theorem Statement (Steiner's Theorem)

The moment of inertia of any plane area about an axis parallel to its centroidal axis is equal to the moment of inertia about the centroidal axis plus the product of the area and the square of the perpendicular distance between the two axes:

$$I_x = I_{xc} + Ad^2$$

where I_{xc} is centroidal MOI, A is total cross-sectional area, and d is perpendicular distance.

Parallel Axis Theorem: Statement & Geometry II



Derivation Step-by-Step

Let an arbitrary plane area A have its centroid at G. Let x_c be the centroidal axis and x be a parallel axis at distance d:

1. Distance of element dA from parallel axis x: $y = y' + d$.
2. Moment of inertia about axis x:
$$I_x = \int y^2 \, dA = \int (y' + d)^2 \, dA$$
3. Expanding the algebraic square term:
$$I_x = \int (y')^2 \, dA + 2d \int y' \, dA + d^2 \int dA$$

Evaluation of Integral Terms

- **Term 1**: $\int (y')^2 \, dA = I_{xc}$ (Centroidal Moment of Inertia).
- **Term 2**: $\int y' \, dA = \bar{y}' A = 0$ (First moment of area about centroid is zero).
- **Term 3**: $d^2 \int dA = Ad^2$ (Area times square of distance).
- **Final Result**: $I_x = I_{xc} + Ad^2 \quad \blacksquare$

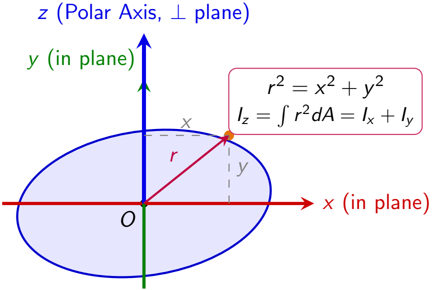
Theorem Statement

The moment of inertia of a flat 2D plane lamina about an axis perpendicular to its plane (polar axis z) passing through origin O equals the sum of moments of inertia about two mutually perpendicular axes (x and y) in the plane intersecting at O:

$$I_z = I_x + I_y \quad \text{or} \quad J_O = I_{xx} + I_{yy}$$

Critical Constraints & Validity

- Applicable **ONLY** to 2D thin planar laminas (plates, disks, cross-sections).
- **NOT valid** for 3D solid bodies (spheres, cylinders, block structures).
- All three axes (x, y, z) must intersect at a single common point O.



z (Polar Axis, $\perp$ plane)
y (in plane)
$r^2 = x^2 + y^2$
$I_z = \int r^2 dA = I_x + I_y$
x
r
y
x (in plane)
O

Cross-Sectional Properties of Engineering Profiles

Real engineering structural designs utilize standard rolled steel sections and geometric profiles with documented area moments of inertia:

Table: Real Engineering Data for Standard Cross-Sections

Structural Profile	**Area** $A\,(\mathbf{mm^2})$	**Centroidal** $I_{xc}\,(\mathbf{mm^4})$	**Centroidal** $I_{yc}\,(\mathbf{mm^4})$	**Polar MOI** $J_z\,(\mathbf{mm^4})$
Solid Rectangle (100×200 mm)	20,000	6.67×10^7	1.67×10^7	8.33×10^7
Solid Circle ($D = 100$ mm)	7,854	4.91×10^6	4.91×10^6	9.82×10^6
Hollow Pipe ($D_o = 100, D_i = 80$ mm)	2,827	2.89×10^6	2.89×10^6	5.78×10^6
ISMB 300 (Medium I-Beam)	5,626	8.60×10^7	4.54×10^6	9.06×10^7
ISMC 200 (Medium Channel)	2,821	1.82×10^7	1.40×10^6	1.96×10^7

Worked Example: T-Section Moment of Inertia

Calculate the moment of inertia I_x about the neutral centroidal axis of a structural T-section:

- **Flange**: $150\,\text{mm} \times 20\,\text{mm}$ (Top component)
- **Web**: $20\,\text{mm} \times 180\,\text{mm}$ (Vertical component)

Step-by-Step Calculation

1. **Component Areas**: $A_1 = 3000\,\text{mm}^2$, $A_2 = 3600\,\text{mm}^2$, $A_{total} = 6600\,\text{mm}^2$.
2. **Centroidal Height** ($\bar{y}$ from base):
$$\bar{y} = \frac{(3000 \times 190) + (3600 \times 90)}{6600} = \frac{894,000}{6600} = 135.45\,\text{mm}$$
3. **Axis Offsets**: $d_1 = 54.55\,\text{mm}, \quad d_2 = 45.45\,\text{mm}.$
4. **Parallel Axis Application**:
$$I_{xc} = \sum \left(I_{xci} + A_i d_i^2\right)$$
$$I_{xc} = (0.10 \times 10^6 + 8.93 \times 10^6) + (9.72 \times 10^6 + 7.44 \times 10^6)$$
$$I_{xc} = 26.19 \times 10^6\,\text{mm}^4$$

Worked Example: Polar MOI of Hollow Shaft

Determine the Polar Moment of Inertia (J_z) and Polar Radius of Gyration (k_z) for a hollow drive shaft with outer diameter $D_o = 120\,\text{mm}$ and inner diameter $D_i = 80\,\text{mm}$.

Mathematical Calculation

- **Planar Centroidal MOI** ($I_x = I_y$):

$$I_x = I_y = \frac{\pi}{64}(D_o^4 - D_i^4) = \frac{\pi}{64}(120^4 - 80^4) = 8.168 \times 10^6\,\text{mm}^4$$

- **Perpendicular Axis Theorem for Polar MOI** (J_z):

$$J_z = I_x + I_y = 2 \times (8.168 \times 10^6\,\text{mm}^4) = 16.336 \times 10^6\,\text{mm}^4$$

- **Polar Radius of Gyration** (k_z):

$$A = \frac{\pi}{4}(120^2 - 80^2) = 6283.19\,\text{mm}^2$$

$$k_z = \sqrt{\frac{J_z}{A}} = \sqrt{\frac{16.336 \times 10^6}{6283.19}} = 50.99\,\text{mm}$$

Table: Comparative Properties of Shaft Planar vs. Polar Inertia

Axis Type	**Symbol**	**Value (mm^4)**	**Engineering Significance**
Planar X-Axis	I_x	8.17×10^6	Resists structural vertical bending
Planar Y-Axis	I_y	8.17×10^6	Resists structural horizontal bending
Polar Z-Axis	J_z	16.34×10^6	Resists shaft torsional shear deformation

Common Engineering Pitfalls to Avoid

- **Non-Centroidal Axis Shift**: Shifting directly between two non-centroidal parallel axes using $I_2 = I_1 + Ad^2$ is **INCORRECT**. You must shift to the centroid first ($I_{xc} = I_1 - Ad_1^2$), then to the new axis ($I_2 = I_{xc} + Ad_2^2$).
- **Dimensional Validity**: Applying the Perpendicular Axis Theorem ($I_z = I_x + I_y$) to 3D bodies (e.g. cylinders, spheres) leads to invalid results.
- **Distance Measurement**: d in Parallel Axis Theorem is strictly the *perpendicular distance* between parallel axes, not oblique distance.

Practical Engineering Applications

- **Beam Cross-Section Optimization**: Placing flanges away from the centroidal axis maximizes Ad^2, producing high strength I-beams with minimal weight.
- **Torsional Shaft Design**: Maximizing polar moment of inertia J_z increases torque capacity per unit material mass in transmission shafts.

- **Definition of Moment of Inertia (Second Moment of Area):** A geometric property of a structural cross-section that measures its resistance to bending deformation and deflection about a specified axis. For an elemental area dA at distance y from axis $X - X$:

$$I_{XX} = \int y^2 \, dA$$

- **Units and Dimensionality:** Expressed in mm^4, cm^4, or m^4.
- **Radius of Gyration (k or r):** The distance from an axis at which the entire area could be concentrated without altering its moment of inertia:

$$k_{xx} = \sqrt{\frac{I_{XX}}{A}}, \qquad k_{yy} = \sqrt{\frac{I_{YY}}{A}}$$

- **Parallel Axis Theorem:** The moment of inertia of any area about an axis AB parallel to its centroidal axis G equals the centroidal moment of inertia I_G plus the product of area A and the square of perpendicular distance h between axes:

$$I_{AB} = I_G + Ah^2$$

- **Perpendicular Axis Theorem:** For a planar laminar area in the XY plane, the polar moment of inertia about Z-axis is:

$$J_z = I_{ZZ} = I_{XX} + I_{YY}$$

Table: Moment of Inertia Formulas for Solid Geometric Sections

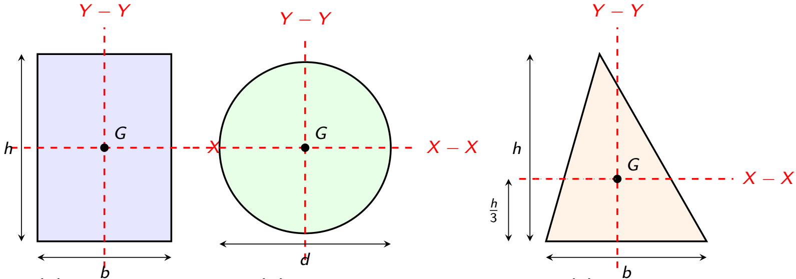
Geometric Shape	**Area** (A)	**Centroidal Axis** (I_{xc}, I_{yc})	**Base / Reference Axis** (I_{base})
Solid Rectangle	$b \times h$	$I_{xc} = \frac{bh^3}{12}$, $I_{yc} = \frac{hb^3}{12}$	$I_{base} = \frac{bh^3}{3}$
Solid Square	a^2	$I_{xc} = I_{yc} = \frac{a^4}{12}$	$I_{base} = \frac{a^4}{3}$
Solid Circle	$\frac{\pi d^2}{4}$	$I_{xc} = I_{yc} = \frac{\pi d^4}{64}$	Polar $J = \frac{\pi d^4}{32}$
Right / Isosceles Triangle	$\frac{1}{2}bh$	$I_{xc} = \frac{bh^3}{36}$	$I_{base} = \frac{bh^3}{12}$, $I_{apex} = \frac{bh^3}{4}$

- **Structural Insight:** Bending resistance scales with the third power of depth (h^3), prioritizing deep vertical orientations over wide horizontal orientations.
- **Symmetry Effect:** Circular and square sections possess equal principal moments of inertia ($I_{xc} = I_{yc}$).

Table: Moment of Inertia Formulas for Hollow and Sub-Sections

Cross-Section Type	**Dimensions**	**Centroidal Moment of Inertia (I_{xc})**	**Polar / Base Axis (J / I_{base})**
Hollow Rectangle	Outer $B \times H$, Inner $b \times h$	$I_{xc} = \frac{BH^3 - bh^3}{12}$	$I_{yc} = \frac{HB^3 - hb^3}{12}$
Hollow Square	Outer side A, Inner side a	$I_{xc} = I_{yc} = \frac{A^4 - a^4}{12}$	$I_{base} = \frac{A^4 - a^4}{3}$
Hollow Circle (Pipe)	Outer dia D, Inner dia d	$I_{xc} = I_{yc} = \frac{\pi(D^4 - d^4)}{64}$	$J = \frac{\pi(D^4 - d^4)}{32}$
Semi-Circle	Radius R, $\bar{y} = \frac{4R}{3\pi}$	$I_{xc} \approx 0.1098R^4$	$I_{base} = \frac{\pi R^4}{8}$
Quarter-Circle	Radius R, $\bar{y} = \frac{4R}{3\pi}$	$I_{xc} \approx 0.0549R^4$	$I_{base} = \frac{\pi R^4}{16}$

- **Material Efficiency:** Hollow sections remove material near the neutral axis where bending stress is minimal, maximizing strength-to-weight ratio.



(a) Rectangle **(b) Circle** **(c) Triangle**

- **Step 1: Datum Reference Axes Selection** Establish baseline axes, typically along the bottom-most (X_{ref}) and left-most (Y_{ref}) outer boundaries.
- **Step 2: Geometric Component Decomposition** Divide the composite shape into n simple geometries (rectangles, triangles, circles). Assign negative area values for hollow cut-outs.
- **Step 3: Centroidal Position Computation**

$$\bar{x} = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i}, \qquad \bar{y} = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i}$$

- **Step 4: Parallel Axis Offset Distances** Calculate distance from each component centroid to overall composite neutral axis:

$$h_{yi} = |y_i - \bar{y}|, \qquad h_{xi} = |x_i - \bar{x}|$$

- **Step 5: Parallel Axis Theorem Summation**

$$I_{XX} = \sum_{i=1}^{n} \left(I_{Gi,x} + A_i h_{yi}^2 \right), \qquad I_{YY} = \sum_{i=1}^{n} \left(I_{Gi,y} + A_i h_{xi}^2 \right)$$

- **Symmetrical I-Section:** Top and bottom flanges are identical. Centroid sits at exact geometric mid-depth ($\bar{y} = H/2$).
- **Asymmetrical I-Section:** Flanges differ in dimensions. Centroid shifts toward the larger flange, requiring explicit calculation of $\bar{y}$.

Table: Real Engineering Data for Standard Indian Standard Beams (ISMB & ISWB)

Designation	**Depth H (mm)**	**Flange B (mm)**	**Web t_w (mm)**	**Area A (cm^2)**	I_{XX} **(cm^4)**	I_{YY} **(cm^4)**	r_{XX} **(cm)**
ISMB 150	150	80	4.8	19.00	773.1	46.2	6.38
ISMB 250	250	125	6.9	47.55	5131.6	334.5	10.39
ISMB 300	300	140	7.5	56.26	8603.6	453.9	12.37
ISWB 400	400	200	8.6	85.01	23426.7	1388.0	16.61

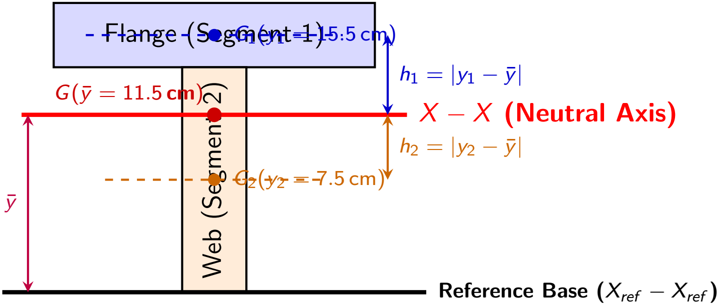
- **T-Section Behavior:** Symmetrical about vertical $Y - Y$ axis ($\bar{x} = B/2$); asymmetrical about horizontal $X - X$ axis.
- **Channel Section (ISMC) Behavior:** Symmetrical about horizontal $X - X$ axis ($\bar{y} = H/2$); asymmetrical about vertical $Y - Y$ axis ($\bar{x} = C_x$).

Table: Real Structural Engineering Properties of Standard Channels and Tees

Profile Designation	**Area** A **(cm^2)**	I_{XX} **(cm^4)**	I_{YY} **(cm^4)**	**Centroid Distance** C_x/C_y **(cm)**	r_{XX} **(cm)**
ISMC 150	20.88	778.3	102.3	$C_x = 2.22$	6.11
ISMC 200	28.21	1819.3	140.4	$C_x = 2.17$	8.03
ISMC 300	45.64	6362.6	310.8	$C_x = 2.36$	11.81
ISNT 150	22.50	482.5	231.0	$C_y = 4.18$	4.63

- **Indian Standard Angles (ISA):**
  - **Equal Angles (e.g., ISA $100 \times 100 \times 10$):** Both legs equal. $\bar{x} = \bar{y} = C_x = C_y$.
  - **Unequal Angles (e.g., ISA $150 \times 110 \times 12$):** Legs unequal. $\bar{x} \neq \bar{y}$; requires principal axis evaluation $(U - U,\ V - V)$.
- **Built-Up Sections:**
  - Structural sections augmented with cover plates (e.g., ISMB 300 with top and bottom $200\,\text{mm} \times 12\,\text{mm}$ plates).
  - Used when standard rolled shapes lack sufficient moment of inertia for heavy bending loads.
  - Calculation relies on Parallel Axis Theorem about composite centroidal axis:

$$I_{XX,total} = I_{XX,beam} + 2\left[I_{G,plate} + A_{plate} h_{plate}^2\right]$$


Flange (Segment 1)
$G_1(y_1 = 15.5\text{ cm})$
$h_1 = |y_1 - \bar{y}|$
$G(\bar{y} = 11.5\text{ cm})$
$X - X$ **(Neutral Axis)**
$h_2 = |y_2 - \bar{y}|$
$G_2(y_2 = 7.5\text{ cm})$
Web (Segment 2)
$\bar{y}$
Reference Base $(X_{ref} - X_{ref})$

Problem Statement

Calculate the Centroid $\bar{y}$ and Centroidal Moment of Inertia I_{XX} for a structural T-section with a top flange of 150 mm × 10 mm and a vertical web of 150 mm × 10 mm.

Table: Step-by-Step Tabular Computation of Centroid and Moment of Inertia

Component	A_i (mm^2)	y_i (mm)	$A_i y_i$ (mm^3)	I_{Gi} (mm^4)	$h_i = \lvert y_i - \bar{y} \rvert$ (mm)	$A_i h_i^2$ (mm^4)	I_{XXi} (mm^4)
Flange (1)	1500	155	232,500	12,500	40	2,400,000	2,412,500
Web (2)	1500	75	112,500	2,812,500	40	2,400,000	5,212,500
Total	**3000**	–	**345,000**	–	–	–	**7,625,000**

- **Centroid Calculation:** $\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{345{,}000\,\text{mm}^3}{3000\,\text{mm}^2} = 115\,\text{mm}$ from base.
- **Total Moment of Inertia:** $I_{XX} = \sum I_{XXi} = 7{,}625{,}000\,\text{mm}^4 = 7.625 \times 10^6\,\text{mm}^4$.
- **Radius of Gyration:** $k_{xx} = \sqrt{\frac{I_{XX}}{A}} = \sqrt{\frac{7{,}625{,}000}{3000}} = 50.41\,\text{mm}$.

Definition of Torsion

- **Torsion** refers to the twisting of a straight structural member or machine shaft when acted upon by moments (or couples) that tend to produce rotation about its longitudinal axis.
- The external moment producing torsion is termed **Torque**, **Turning Moment**, or **Twisting Moment** (T).
- **Standard SI Units**: Newton-meters ($\mathrm{N \cdot m}$) or Kilonewton-meters ($\mathrm{kN \cdot m}$).

Mechanical Action and Stress Generation

- When a shaft is subjected to equal and opposite end torques, cross-sections rotate relative to one another about the central axis.
- This angular displacement creates **torsional shear stresses** (τ) acting on the transverse plane of the cross-section.
- By complementary shear action, equal shear stresses also develop on longitudinal planes parallel to the axis of the shaft.

Practical Engineering Applications

- Drive shafts and propeller shafts in automotive powertrains transmitting mechanical power from engines to wheels.
- Turbine-generator rotor shafts in thermal, nuclear, and hydroelectric power plants.
- Machine tool spindles, drill bits, screw drivers, and helicopter rotor drives.

Fundamental Assumptions (Saint-Venant's Torsion Theory)

To establish the governing equations for pure torsion in circular members, the following engineering assumptions are made:

1. **Material Homogeneity and Isotropy**: The shaft material is uniform throughout and exhibits identical elastic properties in all directions.
2. **Elastic Linearity**: Material obeys Hooke's Law in shear ($\tau = G\gamma$), operating strictly within its elastic limit.
3. **Planar Cross-Sections Remain Plane**: Cross-sections perpendicular to the longitudinal axis remain plane and do not warp after twisting (valid strictly for circular cross-sections).

Fundamental Assumptions Cont.

5. **Uniform Twist Angle**: The angle of twist per unit length $(d\theta/dx)$ is constant along the entire length of a uniform shaft.
6. **Constant Shaft Dimensions**: Axial spacing between adjacent cross-sections remains constant; no axial elongation or contraction occurs.

Kinematic Shear Strain Relationship

Consider a circular shaft of length L and radius R subjected to torque T:

- A longitudinal element AA' at radial distance r distorts by shear strain angle γ.
- Arc length on cross-section: $s = r \cdot \theta = L \cdot \gamma \implies \gamma(r) = \frac{r \cdot \theta}{L}$

Constitutive and Equilibrium Relations

- Applying Hooke's Law ($\tau = G \cdot \gamma$):

$$\tau(r) = \frac{G \cdot r \cdot \theta}{L} \quad \Longrightarrow \quad \frac{\tau}{r} = \frac{G\theta}{L}$$

- Resisting torque T is the integral of elemental forces ($dF = \tau \, dA$) over the cross-sectional area:

$$T = \int_A r \cdot \tau(r) \, dA = \int_A r \left(\frac{G\theta}{L} r \right) dA = \frac{G\theta}{L} \int_A r^2 \, dA = \frac{G\theta}{L} \cdot J$$

Unified Governing Torsion Equation

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

where T = Resisting Torque, J = Polar Moment of Inertia, τ = Shear stress at radius r, G = Modulus of Rigidity, θ = Angle of twist (rad), L = Length.

Polar Moment of Inertia (J)

J measures a cross-section's geometric resistance to torsional deformation:

$$J = \int_A r^2 \, dA = I_x + I_y \quad \text{(Perpendicular Axis Theorem)}$$

- **Solid Circular Shaft** (diameter D):

$$J_{\text{solid}} = \frac{\pi D^4}{32} \approx 0.0982 D^4$$

- **Hollow Circular Shaft** (outer diameter D_o, inner diameter D_i):

$$J_{\text{hollow}} = \frac{\pi (D_o^4 - D_i^4)}{32}$$

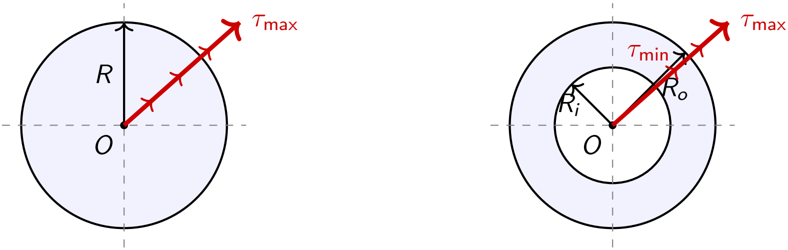
Polar Section Modulus (Z_p)

Defined as $Z_p = \frac{J}{R_{\text{max}}}$, representing torsional strength capacity:

- **Solid Shaft**: $Z_{p,\text{solid}} = \frac{\pi D^3}{16} \approx 0.1963 D^3$
- **Hollow Shaft**: $Z_{p,\text{hollow}} = \frac{\pi (D_o^4 - D_i^4)}{16 D_o}$
- **Maximum Shear Stress**: $\tau_{\text{max}} = \frac{T}{Z_p} \implies T_{\text{max}} = \tau_{\text{allow}} \cdot Z_p$

Stress Variation Characteristics

- Shear stress varies **linearly** with radial distance r from zero at the neutral center axis ($r = 0$) to maximum τ_{max} at outer surface ($r = R$).
- In hollow shafts, minimum shear stress τ_{min} occurs at inner radius $r = R_i$, and maximum stress τ_{max} occurs at outer radius $r = R_o$.



(a) Solid Shaft
$\tau = 0$ at $r = 0$

(b) Hollow Shaft
$\tau_{min} > 0$ at $r = R_i$

Angle of Twist (θ)

The total angular rotation between two cross-sections separated by distance L:

$$\theta = \frac{T \cdot L}{G \cdot J} \quad \text{(in radians)}$$

To convert radians to degrees:

$$\theta^\circ = \left(\frac{180}{\pi}\right)\frac{T \cdot L}{G \cdot J} \approx 57.2958 \cdot \frac{T \cdot L}{G \cdot J}$$

Torsional Rigidity ($G \cdot J$)

- Product of Shear Modulus (G) and Polar Moment of Inertia (J).
- Represents the torque required to produce a unit angle of twist per unit length ($L = 1$ m, $\theta = 1$ rad).
- **SI Unit**: $\mathrm{N} \cdot \mathrm{m}^2$.

Torsional Stiffness (K_t) and Flexibility (f_t)

- **Torsional Stiffness**: Torque required per unit angle of twist for length L:

$$K_t = \frac{T}{\theta} = \frac{G \cdot J}{L} \quad [\mathrm{N \cdot m/rad}]$$

- **Torsional Flexibility**: Reciprocal of torsional stiffness:

$$f_t = \frac{1}{K_t} = \frac{L}{G \cdot J} \quad [\mathrm{rad/(N \cdot m)}]$$

Torsional Strength Criterion

The **strength of a shaft** is governed by the maximum torque it can safely transmit without exceeding the allowable shear stress $[\tau_{\text{allow}}]$:

$$T_{\text{safe}} = \tau_{\text{allow}} \cdot Z_p$$

Power Transmission Mechanics

Torque developed by a motor or engine transmitting mechanical power P at rotational speed N (rpm):

$$P = T \cdot \omega = T\left(\frac{2\pi N}{60}\right) \quad \Longrightarrow \quad T = \frac{60P}{2\pi N}$$

where P is in Watts (W), N in rpm, and T in $\text{N} \cdot \text{m}$.

Why Hollow Shafts are Superior in Structural Efficiency

- Material near the central axis of a solid shaft experiences low shear stress ($\tau \to 0$), making inefficient use of material weight.
- By moving mass away from the neutral axis into a hollow configuration, J and Z_p increase substantially for the same total cross-sectional area.
- **Weight Savings**: A hollow shaft with ratio $k = D_i/D_o = 0.75$ delivers equal torque capacity as a solid shaft while achieving **over** 45% **weight reduction**.

Engineering Shaft Material Properties

Real engineering values for common shaft materials used in mechanical power transmission design:

Table: Mechanical & Torsional Properties of Standard Shaft Engineering Materials

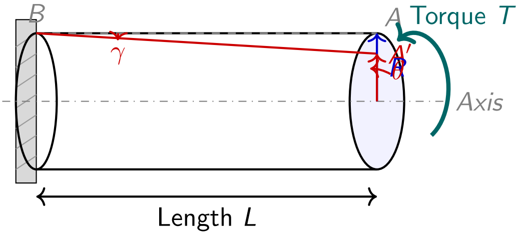
Material Specification	**Density** ρ (kg/m^3)	**Shear Modulus** G (GPa)	**Yield Shear** τ_y (MPa)	**Allow. Shear** $[\tau]$ (MPa)
AISI 1045 Carbon Steel	7850	80.0	185	95–100
AISI 4340 Alloy Steel	7850	80.0	490	240–250
Alloy Steel 4140 (Q&T)	7850	79.3	415	210
Aluminum 7075-T6	2810	26.9	190	95–100
Titanium Ti-6Al-4V	4430	44.0	330	165–180
Brass C36000 (Free-Cutting)	8500	39.0	120	60–65

Table: Design Comparison: Solid vs. Hollow Shaft ($T = 2.5$ kN $\cdot$ m, $[\tau] = 80$ MPa)

Shaft Configuration	**Dimensions (mm)**	**Area (cm^2)**	**Mass (kg/m)**	**Rigidity GJ (kN $\cdot$ m^2)**
Solid Circular Shaft	$D = 54.2$	23.07	18.11	67.68
Hollow Shaft ($k = 0.75$)	$D_o = 60.0, D_i = 45.0$	12.37	9.71	100.80
Performance Gain	–	**-46.4% Area**	**-46.4% Mass**	**+48.9% Rigidity**

Kinematics of Torsional Deformation

Schematic visualization of a circular shaft fixed at one end and twisted by torque T:



Dual Shaft Design Criteria Summary

A transmission shaft must satisfy two independent design conditions:

1. **Strength Criterion**:

$$\tau_{\max} = \frac{T}{Z_p} \leq [\tau_{\text{allow}}] = \frac{\tau_y}{\text{FOS}}$$

2. **Stiffness Criterion** (Twist Limit):

$$\theta = \frac{T \cdot L}{G \cdot J} \leq [\theta_{\text{allow}}] \quad \text{(typically } 1^\circ \text{ per } 20D \text{ length)}$$

GTU Course: DI03000201 – Mechanics of Solids / Strength of Materials

Unit: Torsion
Topic: Torque, Angle of Twist, Shear Stress, and Shaft Strength

- **Definition of Torsion:** Torsion refers to the twisting of a structural member or machine component when it is subjected to equal and opposite couples (torques) acting about its longitudinal axis.
- **Key Terminology:**
  - **Torque (T) / Turning Moment / Twisting Moment:** The product of a tangential force and its perpendicular distance from the axis of rotation ($\mathrm{N} \cdot \mathrm{m}$).

- **Angle of Twist (θ):** The angle through which one cross-section of a shaft rotates relative to another section separated by length L (radians).
- **Shear Stress (τ):** Internal resisting force per unit area acting tangentially to the cross-sectional plane (N/m^2 or MPa).

- **Engineering Applications:**
  - Automotive drive shafts (propeller shafts) transmitting torque from engine to wheels.
  - Machine tool spindles (lathe, milling machine drives).
  - Steam turbine and electric motor rotor shafts.

Fundamental Assumptions

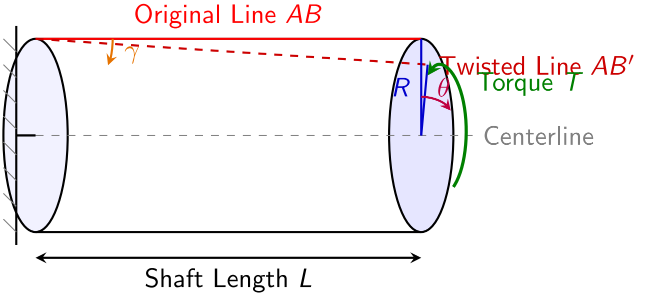
To derive the classic torsion equation for circular shafts, the following assumptions are made:

1. **Material Homogeneity and Isotropy:** The material of the shaft is uniform throughout (homogeneous) and possesses identical mechanical properties in all directions (isotropic).
2. **Hooke's Law Validity:** The shear stress developed is directly proportional to the shear strain, remaining strictly within the elastic limit ($\tau = G \cdot \gamma$).
3. **Planar Cross-Sections:** Cross-sections of the shaft that are plane before twisting remain plane after twisting (no warping or out-of-plane distortion occurs).
4. **Straight Radial Lines:** Radial lines drawn on the cross-section remain straight and radial after deformation.

5. **Uniform Twist along Length:** The angle of twist per unit length is uniform along the entire longitudinal axis of the shaft.
6. **Circular Cross-Section Symmetry:** The shaft is of uniform circular cross-section (solid or hollow) throughout its length.

Relationship Between Shear Strain (γ) and Angle of Twist (θ)

Consider a circular shaft of radius R and length L subjected to a twisting moment T.



- Arc displacement on outer surface: Arc $BB' = R \cdot \theta = L \cdot \gamma$

- Shear strain at outer surface: $\gamma = \frac{R\theta}{L}$
- By Hooke's Law ($\tau = G\gamma$): $\tau_{\text{max}} = \frac{GR\theta}{L} \implies \frac{\tau}{r} = \frac{G\theta}{L}$

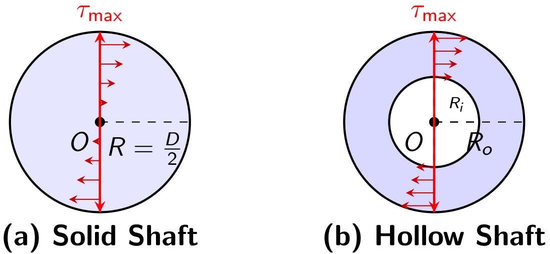
Governing Torsion Formula

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

- **Parameters and Standard SI Units:**
  - T: Applied Twisting Torque ($\mathrm{N \cdot m}$)
  - J: Polar Moment of Inertia of the cross-section (m^4)
  - τ: Shear stress developed at radial distance r ($\mathrm{N/m}^2$ or Pa)
  - r: Radial distance from central axis (m)
  - G: Shear Modulus / Modulus of Rigidity of material ($\mathrm{N/m}^2$ or Pa)
  - θ: Total angle of twist (radians)
  - L: Length of the shaft subjected to torsion (m)
- **Shear Stress Variation Across Cross-Section:**
  - Shear stress varies **linearly** with radial distance r.
  - At the center of shaft ($r = 0$), shear stress is zero ($\tau = 0$).

- At the outer surface ($r = R = D/2$), shear stress reaches maximum value ($\tau = \tau_{\text{max}}$).

Geometric Properties of Solid and Hollow Circular Shafts



- **Solid Shaft (D):**

$$J = \int r^2\, dA = \frac{\pi D^4}{32}, \qquad Z_p = \frac{J}{R} = \frac{\pi D^3}{16}$$

- **Hollow Shaft (D_o, D_i):**

$$J = \frac{\pi(D_o^4 - D_i^4)}{32}, \qquad Z_p = \frac{J}{R_o} = \frac{\pi(D_o^4 - D_i^4)}{16 D_o}$$

Definitions and Mathematical Expressions

- **1. Torsional Rigidity (GJ):**
  - Product of Shear Modulus (G) and Polar Moment of Inertia (J).
  - Represents the torque required to produce a unit angle of twist per unit length of shaft ($\frac{\theta}{L} = 1$ rad/m).
  - SI Unit: $\mathrm{N} \cdot \mathrm{m}^2$.
- **2. Torsional Stiffness / Torsional Spring Constant (k_t):**
  - Torque required to produce a unit total angle of twist ($\theta = 1$ rad) over length L.

$$k_t = \frac{T}{\theta} = \frac{GJ}{L}$$

  - SI Unit: $\mathrm{N} \cdot \mathrm{m}$ / rad.
- **3. Torsional Flexibility (f_t):**

- Reciprocal of torsional stiffness; angle of twist produced per unit applied torque.

$$f_t = \frac{1}{k_t} = \frac{\theta}{T} = \frac{L}{GJ}$$

- SI Unit: rad / (N · m).

Torque Capacity and Power Calculations

- **Torsional Strength (Torque Capacity T_{max}):** Maximum torque a shaft can safely transmit without exceeding allowable shear stress τ_{allow}:

$$T_{\text{max}} = \tau_{\text{allow}} \cdot Z_p$$

  - For Solid Shaft: $T_{\text{max}} = \tau_{\text{allow}} \cdot \left(\frac{\pi D^3}{16}\right)$
  - For Hollow Shaft: $T_{\text{max}} = \tau_{\text{allow}} \cdot \left(\frac{\pi(D_o^4 - D_i^4)}{16 D_o}\right)$

- **Power Transmission in Shafts:** Power P transmitted by a shaft rotating at N revolutions per minute (rpm) with mean torque T:

$$\omega = \frac{2\pi N}{60} \quad \text{(rad/s)}$$

$$P = T \cdot \omega = \frac{2\pi NT}{60} \quad \text{(Watts)}$$

- $1\ \text{kW} = 10^3\ \text{W}, \quad 1\ \text{MW} = 10^6\ \text{W}$
- $1\ \text{HP (Metric)} = 735.5\ \text{W}$

Standard Mechanical Properties for Shaft Design

Real engineering design parameters at room temperature (20°C).

Material Specification	**Shear Modulus** G (GPa)	**Yield Shear Stress** τ_y (MPa)	**Allow. Shear Stress** τ_{allow} (MPa)	**Density** ρ (kg/m^3)
Structural Steel (AISI 1020)	79.3	145	60	7850
Alloy Steel (AISI 4140 Qt)	80.0	380	120	7850
Stainless Steel (AISI 304)	75.0	130	55	8000
Aluminum Alloy (6061-T6)	26.0	165	40	2700
Brass (C36000 Free-Cutting)	39.0	120	35	8500
Titanium Alloy (Ti-6Al-4V)	44.0	330	110	4430

Table: Engineering material properties for torsional design calculations.

Comparative Analysis for Same Torque Capacity ($T = 5.0$ kN $\cdot$ m, $\tau_{\text{allow}} = 60$ MPa)

Engineering evaluation demonstrating mass reduction and structural efficiency of hollow shafts.

Design Parameter	**Solid Shaft**	**Hollow Shaft ($K = \frac{D_i}{D_o} = 0.6$)**	**Comparison / Saving**
Outer Diameter (D_o)	75.1 mm	77.8 mm	+3.6% diameter
Inner Diameter (D_i)	0 mm	46.7 mm	Internal cavity
Cross-Area (A)	4429 mm^2	3043 mm^2	**31.3**% area reduction
Polar Modulus (Z_p)	83.33 cm^3	83.33 cm^3	Equal torque capacity
Mass per meter (kg/m)	34.77 kg/m	23.89 kg/m	**31.3**% mass saving
Stiffness (GJ/L)	Baseline	+14.2%	Higher torsional rigidity

Table: Quantitative comparison showing weight efficiency of hollow shafts.

Master Formula Summary for Torsion

- **Torsion Master Equation:** $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$
- **Solid Shaft (D):** $J = \frac{\pi D^4}{32}, \quad Z_p = \frac{\pi D^3}{16}, \quad T = \tau_{\text{allow}} \cdot \frac{\pi D^3}{16}$
- **Hollow Shaft (D_o, D_i):** $J = \frac{\pi(D_o^4 - D_i^4)}{32}, \quad Z_p = \frac{\pi(D_o^4 - D_i^4)}{16 D_o}$
- **Torsional Stiffness:** $k_t = \frac{GJ}{L} \quad (\text{N} \cdot \text{m/rad})$
- **Power Transmitted:** $P = \frac{2\pi N T}{60} \quad (\text{Watts})$

GTU Exam Review Questions

1. Derive the torsion equation $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ stating all assumptions clearly.
2. Compare the weight efficiency of a solid shaft and a hollow shaft of the same material and length transmitting equal torque.

- **GTU Course Code:** DI03000201 – Mechanics of Solids
- **Unit 5:** Torsion in Shafts
- **Lecture Focus:** Section 5.2 – Governing Torsion Equation and Practical Engineering Applications

The Fundamental Torsion Equation

For a circular shaft (solid or hollow) subjected to a pure twisting moment (torque) T, the governing torsion equation is given by:

$$\frac{T}{J_p} = \frac{\tau}{r} = \frac{G \cdot \theta}{L}$$

Definition of Nomenclature and SI Units

- T = Twisting moment or Applied Torque ($N \cdot m$ or $N \cdot mm$)
- J_p = Polar Moment of Inertia of the cross-section (mm^4 or m^4)
- τ = Shear stress at radial distance r from axis (N/mm^2 or MPa)
- r = Radial distance of point under consideration from central axis (mm)
- G = Shear Modulus or Modulus of Rigidity of shaft material (N/mm^2 or GPa)
- θ = Angle of twist over length L (radians)
- L = Length of shaft subjected to torsion (mm or m)

Polar Moment of Inertia (J_p)

The polar moment of inertia measures a cross-section's resistance to torsional deformation:

$$J_p = I_{xx} + I_{yy}$$

- **Solid Circular Shaft (Diameter D):**

$$J_p = \frac{\pi}{32} D^4$$

- **Hollow Circular Shaft (Outer D_o, Inner D_i):**

$$J_p = \frac{\pi}{32}(D_o^4 - D_i^4)$$

Polar Section Modulus (Z_p)

Polar section modulus defines the maximum torque capacity per unit allowable stress:

$$Z_p = \frac{J_p}{R} = \frac{J_p}{D/2}$$

- **Solid Shaft:** $Z_p = \frac{\pi}{16} D^3 \implies T_{max} = \tau_{allow} \cdot Z_p = \frac{\pi}{16} \tau_{allow} D^3$
- **Hollow Shaft:** $Z_p = \frac{\pi}{16} \left(\frac{D_o^4 - D_i^4}{D_o} \right) \implies T_{max} = \frac{\pi}{16} \tau_{allow} \left(\frac{D_o^4 - D_i^4}{D_o} \right)$

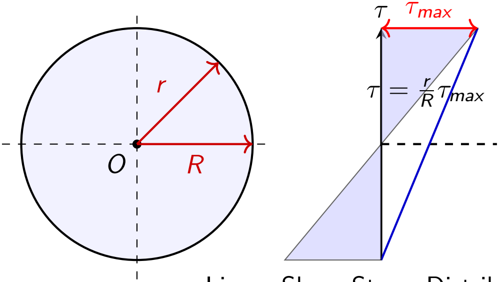
Table: Real Engineering Mechanical Properties of Shaft Materials

Material Specification	**Shear Modulus G (GPa)**	**Allowable Shear Stress τ_{allow} (MPa)**	**Mass Density ρ (kg/m^3)**
Structural Steel (Fe 415)	80.0	60.0 – 75.0	7850
High Strength Steel (Fe 500)	79.0	85.0 – 110.0	7850
Aluminum Alloy (6061-T6)	26.0	45.0 – 55.0	2700
Phosphor Bronze (C51000)	41.5	35.0 – 45.0	8800
Titanium Alloy (Ti-6Al-4V)	44.0	140.0 – 160.0	4430

Torsional Rigidity and Stiffness

- **Torsional Rigidity ($G \cdot J_p$):** Product of shear modulus and polar moment of inertia ($\mathrm{N} \cdot \mathrm{mm}^2$). Represents resistance to twisting.
- **Torsional Stiffness (K_t):** Torque required to produce unit angle of twist (1 rad):

$$K_t = \frac{T}{\theta} = \frac{G \cdot J_p}{L} \quad (\mathrm{N} \cdot \mathrm{m/rad})$$



Linear Shear Stress Distribution

Key Metallurgical & Stress Distribution Characteristics

- Shear stress varies **linearly** from zero at the geometric axis ($r = 0$) to maximum τ_{max} at outer fiber ($r = R$).
- Material near the center operates at low stress levels, making solid shaft centers structurally inefficient.
- Maximum shear strain occurs at outer radius: $\gamma_{max} = \frac{R \cdot \theta}{L}$.

Problem Statement

A solid steel shaft ($G = 80$ GPa) is required to transmit a torque of 12 kN $\cdot$ m. The maximum shear stress is limited to 65 MPa and the permissible angle of twist is not to exceed 1.5° in a length of 3 m. Determine the minimum required shaft diameter D.

Step 1: Diameter Based on Strength Criteria ($\tau_{max} \leq 65$ MPa)

Given: $T = 12 \times 10^6$ N $\cdot$ mm, $\tau_{allow} = 65$ N/mm^2.

$$T = \frac{\pi}{16}\tau_{allow} D^3 \implies 12 \times 10^6 = \frac{\pi}{16}(65)D^3$$

$$D^3 = \frac{16 \times 12 \times 10^6}{65\pi} = 940,248.6 \text{ mm}^3 \implies D_1 = 97.97 \text{ mm}$$

Step 2: Diameter Based on Stiffness Criteria ($\theta \leq 1.5^\circ$)

Convert twist angle to radians: $\theta = 1.5 \times \frac{\pi}{180} = 0.02618$ rad.
Length $L = 3000$ mm, $G = 80,000$ N/mm^2.

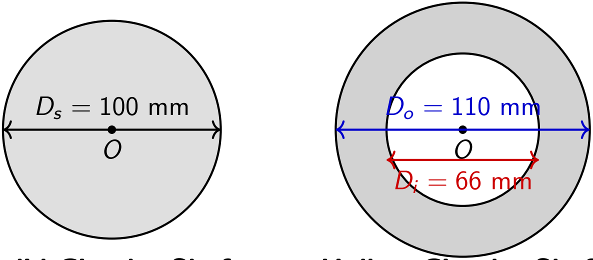
$$\frac{T}{J_p} = \frac{G\theta}{L} \implies J_p = \frac{\pi}{32}D^4 = \frac{T \cdot L}{G \cdot \theta}$$

$$D^4 = \frac{32 \cdot T \cdot L}{\pi \cdot G \cdot \theta} = \frac{32 \times (12 \times 10^6) \times 3000}{\pi \times 80,000 \times 0.02618} = 174,710,653 \text{ mm}^4 \implies D_2 = 114.92 \text{ mm}$$

Final Design Selection

To satisfy both strength and stiffness criteria simultaneously, select the larger value:

$$D = \max(D_1, D_2) = \mathbf{115} \text{ mm} \quad \text{(Standard commercial size)}$$



Solid Circular Shaft
$A_s = 7854\ \text{mm}^2$

Hollow Circular Shaft
$A_h = 6082\ \text{mm}^2$

Table: Comparative Performance Data of Solid vs Hollow Shafts (Equal τ_{max})

Design Parameter	**Solid Shaft**	**Hollow Shaft ($D_i/D_o = 0.6$)**	**Performance Gain**
Outer Diameter (D_o)	100.0 mm	110.0 mm	$+10.0\%$
Cross-Section Area (A)	7854 mm^2	6082 mm^2	$-$**22.56**% Weight
Torque Capacity (T)	11.78 $kN \cdot m$	14.15 $kN \cdot m$	$+$**20.12**% Torque
Specific Strength (T/m)	Baseline (1.00)	$1.55 \times$ Solid	$+$**55.1**% Efficiency

Problem Statement

A hollow shaft of internal diameter equal to 0.6 times its external diameter is to replace a solid shaft of diameter 80 mm to transmit the same torque at the same maximum shear stress. Compare the weights of the two shafts for equal length.

Step 1: Equating Torque Capacity ($T_{hollow} = T_{solid}$)

For equal maximum shear stress τ_{max}:

$$Z_{p,hollow} = Z_{p,solid}$$

$$\frac{\pi}{16}\frac{D_o^4 - D_i^4}{D_o} = \frac{\pi}{16}D_s^3$$

Given $D_i = 0.6D_o$ and $D_s = 80$ mm:

$$\frac{D_o^4 - (0.6D_o)^4}{D_o} = 80^3 \implies D_o^3(1 - 0.1296) = 512,000$$

$$0.8704 \cdot D_o^3 = 512,000 \implies D_o^3 = 588,235.3 \implies D_o = 83.79 \text{ mm}$$

$$D_i = 0.6 \times 83.79 = 50.27 \text{ mm}$$

Step 2: Weight Ratio Calculation

Since weight is proportional to cross-sectional area for equal length and material density:

$$\frac{W_{hollow}}{W_{solid}} = \frac{A_{hollow}}{A_{solid}} = \frac{\frac{\pi}{4}(D_o^2 - D_i^2)}{\frac{\pi}{4}D_s^2} = \frac{83.79^2 - 50.27^2}{80^2}$$

$$\frac{A_{hollow}}{A_{solid}} = \frac{7020.77 - 2527.07}{6400} = \frac{4493.7}{6400} = 0.7021$$

$$\text{Percentage Weight Saving} = (1 - 0.7021) \times 100\% = \mathbf{29.79\%}$$

Problem Statement

A stepped shaft fixed at one end consists of a steel section ($L_1 = 1.2$ m, $D_1 = 50$ mm, $G_1 = 80$ GPa) and a brass section ($L_2 = 0.8$ m, $D_2 = 40$ mm, $G_2 = 40$ GPa). A twisting moment $T = 800$ N $\cdot$ m is applied at the free end. Calculate the maximum shear stress in each section and total angle of twist at free end.

Step 1: Calculate Maximum Shear Stresses

Applied torque $T = 800 \times 10^3$ N $\cdot$ mm (constant throughout series shaft).

- **Steel Section ($D_1 = 50$ mm):**

$$\tau_1 = \frac{16T}{\pi D_1^3} = \frac{16 \times 800 \times 10^3}{\pi \times 50^3} = \mathbf{32.59} \text{ MPa}$$

- **Brass Section ($D_2 = 40$ mm):**

$$\tau_2 = \frac{16T}{\pi D_2^3} = \frac{16 \times 800 \times 10^3}{\pi \times 40^3} = \mathbf{63.66} \text{ MPa}$$

Step 2: Calculate Total Angle of Twist ($\theta_{total} = \theta_1 + \theta_2$)

Calculate polar moments of inertia:

$$J_{p1} = \frac{\pi}{32}(50)^4 = 613,592 \text{ mm}^4, \quad J_{p2} = \frac{\pi}{32}(40)^4 = 251,327 \text{ mm}^4$$

Calculate individual angles of twist:

$$\theta_1 = \frac{T \cdot L_1}{G_1 \cdot J_{p1}} = \frac{(800 \times 10^3) \times 1200}{(80,000) \times 613,592} = 0.01955 \text{ rad}$$

$$\theta_2 = \frac{T \cdot L_2}{G_2 \cdot J_{p2}} = \frac{(800 \times 10^3) \times 800}{(40,000) \times 251,327} = 0.06366 \text{ rad}$$

$$\theta_{total} = 0.01955 + 0.06366 = 0.08321 \text{ rad} = \mathbf{4.777}^\circ$$

Key Summary Takeaways

- Torsion equation $\frac{T}{J_p} = \frac{\tau}{r} = \frac{G\theta}{L}$ governs circular shaft stress and twist angle.
- Shaft design must evaluate both **Strength** ($\tau_{max} \leq \tau_{allow}$) and **Stiffness** ($\theta \leq \theta_{allow}$).
- Hollow shafts offer up to 30% to 50% weight reduction compared to solid shafts for identical torque transmission.

GTU Examination Practice Problems

1. Write the torsion equation for a solid circular shaft, defining each term with its standard SI units. [3 Marks]
2. Derive the expression for polar section modulus of a hollow circular shaft. [4 Marks]
3. A hollow shaft transmits 300 kW at 200 rpm. Internal diameter is 0.6 times external diameter. Max shear stress is limited to 60 MPa. Find external and internal diameters. [7 Marks]

- **Course Code:** DI03000201 – Mechanics of Solids / Machine Components
- **Unit:** Torsion in Circular Shafts
- **Lecture Focus:** Section 26 – Relationship between Horsepower (H.P.), Torque (Torsion), and Rotational Speed (RPM) with Practical Engineering Numericals

Role of Torsion in Power Transmission

Rotating shafts are the fundamental mechanical components used to transmit power from prime movers (electric motors, internal combustion engines, steam turbines) to driven machinery (pumps, compressors, machine tools, automotive wheels).

Key Engineering Variables

- **Power (P):** Rate of doing mechanical work transmitted by the rotating shaft (W, kW, or hp).
- **Torque (T):** Torsional moment or turning couple acting on the shaft ($\mathrm{N \cdot m}$ or $\mathrm{N \cdot mm}$).
- **Rotational Speed (N):** Speed of shaft rotation in Revolutions Per Minute (RPM).
- **Angular Velocity (ω):** Rotational speed in radians per second (rad/s), where $\omega = \frac{2\pi N}{60}$.

Fundamental Power Equation

Work done by a constant torque T turning through an angle of rotation θ radians is:

$$W = T \cdot \theta \quad (\text{Joules or N} \cdot \text{m})$$

In one complete revolution, angle of rotation $\theta = 2\pi$ rad. For a shaft rotating at N RPM, revolutions per second is $\frac{N}{60}$.

$$\text{Power } P = \frac{\text{Work Done}}{\text{Time}} = T \cdot \left(\frac{2\pi N}{60}\right) = T \cdot \omega \quad (\text{Watts})$$

Standard Unit Formulations

- **Power in Kilowatts (kW):**

$$P(\text{kW}) = \frac{2\pi NT}{60000} = \frac{\pi NT}{30000} \quad (T \text{ in N} \cdot \text{m})$$

- **Power in Metric Horsepower ($\textbf{H.P.}_{\text{metric}}$):**
  ($1 \text{ hp}_{\text{metric}} = 735.5 \text{ W} = 75 \text{ kgf} \cdot \text{m/s}$)

$$\text{H.P.}_{\text{metric}} = \frac{2\pi NT}{4500} \quad (T \text{ in kgf} \cdot \text{m}) \quad \text{or} \quad \text{H.P.} = \frac{P(\text{kW})}{0.7355}$$

- **Power in Imperial Horsepower ($\textbf{H.P.}_{\text{imperial}}$):**
  ($1 \text{ hp}_{\text{imperial}} = 745.7 \text{ W} = 550 \text{ ft} \cdot \text{lbf/s}$)

$$\text{H.P.}_{\text{imperial}} = \frac{P(\text{kW})}{0.7457} \approx \frac{2\pi NT}{60000 \times 0.7457}$$

Distinction Between T_{mean} and T_{max}

The power equation yields the **Mean Torque** (T_{mean}) transmitted over time:

$$T_{\text{mean}} = \frac{60000 \cdot P(\text{kW})}{2\pi N} = \frac{9549.3 \cdot P(\text{kW})}{N} \quad (\text{N} \cdot \text{m})$$

However, real industrial prime movers (IC engines, reciprocating compressors) generate fluctuating torques with peak values T_{max}.

Design Factor for Torque Fluctuation

A torque overload factor or maximum torque multiplier k_t is defined as:

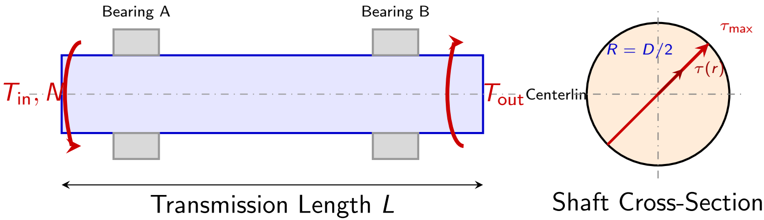
$$T_{\text{max}} = k_t \cdot T_{\text{mean}} \quad (\text{where } k_t \geq 1.0, \text{ typically } 1.15 \text{ to } 1.50)$$

CRITICAL DESIGN RULE: All strength calculations ($\tau_{\text{max}} \leq [\tau]$) and shaft sizing MUST be performed using T_{max}, NOT T_{mean}.

Torsion Equation for Solid and Hollow Shafts

$$\frac{T_{\text{max}}}{J} = \frac{\tau_{\text{max}}}{R} \implies T_{\text{max}} = \tau_{\text{allow}} \cdot Z_p$$

- **Solid Shaft (D):** $Z_p = \frac{\pi D^3}{16} \implies T_{\text{max}} = \frac{\pi}{16}\tau_{\text{allow}} D^3$
- **Hollow Shaft ($D_o, D_i = kD_o$):**
  $Z_p = \frac{\pi D_o^3(1-k^4)}{16} \implies T_{\text{max}} = \frac{\pi}{16}\tau_{\text{allow}} D_o^3(1-k^4)$



Shaft Cross-Section
Linear Shear Stress $\tau(r)$

Physical Interpretation of Diagram

- Power P is delivered at RPM N, generating torsional moment T_{in} at input.
- Shear stress varies linearly from zero at the central axis to maximum τ_{max} at outer radius R.
- Bearings support radial reaction loads while allowing free rotation.

Table: Mechanical Properties & Operational Parameters for Industrial Transmission Shafts

Material Specification	**Shear Modulus** G **(GPa)**	**Allowable Stress** $[\tau]$ **(MPa)**	**Speed Range (RPM)**	**Typical Power Application**
Plain Carbon Steel (AISI 1020 / C20)	79.3	45 – 55	100 – 1500	General Line Shafts ($<$ 50 kW)
Medium Carbon Steel (AISI 1045 / C45)	80.0	60 – 75	300 – 3000	Industrial Gearboxes (50–250 kW)
Alloy Steel (40Cr4Mo2 / AISI 4140)	80.0	80 – 110	500 – 10000	High-Speed Turbines ($>$ 250 kW)
Austenitic Stainless (AISI 304)	75.0	50 – 65	200 – 2500	Marine Pumps & Chemical Mixers
Aluminum Alloy (6061-T6)	26.0	30 – 40	500 – 5000	Aircraft Drive Shafts & Robotics

Design Observations from Engineering Data

- Higher allowable shear stress $[\tau]$ enables significantly reduced shaft diameter for identical power transmission.
- Steel alloys maintain nearly constant shear modulus ($G \approx 75$–80 GPa), keeping torsional stiffness ($\frac{GJ}{L}$) consistent across grades.

Problem Statement

An electric motor transmits 150 hp (111.86 kW) to an industrial blower at 300 RPM. The maximum torque exceeds the mean torque by 30%. Determine the required diameter of a solid steel shaft if the maximum allowable shear stress is limited to 60 MPa.

Step 1: Calculate Power and Mean Torque

Power

$P = 150 \text{ hp} = 150 \times 0.7457 \text{ kW} = 111.855 \text{ kW} = 111,855 \text{ W}.$

$$T_{\text{mean}} = \frac{60 \cdot P}{2\pi N} = \frac{60 \times 111855}{2\pi \times 300} = \frac{6711300}{1884.956} = 3560.46 \text{ N} \cdot \text{m}$$

Step 2: Determine Peak Torque T_{max}

Given $T_{\text{max}} = 1.30 \times T_{\text{mean}}$:

$$T_{\text{max}} = 1.30 \times 3560.46 \text{ N}\cdot\text{m} = 4628.60 \text{ N}\cdot\text{m} = 4.6286 \times 10^6 \text{ N}\cdot\text{mm}$$

Step 3: Calculate Solid Shaft Diameter D

Using strength design formula $T_{\text{max}} = \frac{\pi}{16}\tau_{\text{allow}} D^3$:

$$D^3 = \frac{16 \cdot T_{\text{max}}}{\pi \cdot \tau_{\text{allow}}} = \frac{16 \times 4.6286 \times 10^6}{\pi \times 60} = \frac{74.0576 \times 10^6}{188.4956} = 392,887.8 \text{ mm}^3$$

$$D = \sqrt[3]{392887.8} = 73.24 \text{ mm} \quad \Longrightarrow \text{ Adopt Standard Size: } \mathbf{D} = \mathbf{75} \text{ mm}$$

Problem Statement

A hollow transmission shaft with internal-to-external diameter ratio $k = D_i/D_o = 0.6$ transmits 300 kW at 250 RPM. Allowable shear stress is $[\tau] = 70$ MPa.

1. Determine required external diameter D_o and internal diameter D_i.
2. Calculate percentage weight savings compared to a solid shaft transmitting the same power at the same speed and stress.

Step 1: Calculate Transmitted Torque

$$T = \frac{60000 \cdot P(\text{kW})}{2\pi N} = \frac{60000 \times 300}{2\pi \times 250} = 11,459.16 \text{ N} \cdot \text{m} = 11.459 \times 10^6 \text{ N} \cdot \text{mm}$$

Step 2: Size Hollow Shaft (D_o and D_i)

$$T = \frac{\pi}{16}\tau D_o^3(1-k^4) \implies 11.459 \times 10^6 = \frac{\pi}{16} \times 70 \times D_o^3 \times (1-0.6^4)$$

Since $1 - 0.6^4 = 1 - 0.1296 = 0.8704$:

$$11.459 \times 10^6 = 11.963 \cdot D_o^3 \implies D_o^3 = 957,883 \text{ mm}^3 \implies D_o = 98.57 \text{ mm}$$

$$D_i = 0.6 \times 98.57 = 59.14 \text{ mm}$$

Step 3: Size Equivalent Solid Shaft (D_s) & Compare Mass

For solid shaft:

$D_s^3 = \frac{16T}{\pi\tau} = \frac{16\times11.459\times10^6}{\pi\times70} = 833,744\ \text{mm}^3 \implies D_s = 94.12\ \text{mm}.$

$$\text{Area of Solid Shaft } A_s = \frac{\pi}{4}D_s^2 = \frac{\pi}{4}(94.12)^2 = 6958\ \text{mm}^2$$

$$\text{Area of Hollow Shaft } A_h = \frac{\pi}{4}(D_o^2 - D_i^2) = \frac{\pi}{4}(98.57^2 - 59.14^2) = 4884\ \text{mm}^2$$

$$\text{Weight Savings \%} = \frac{A_s - A_h}{A_s} \times 100\% = \frac{6958 - 4884}{6958} \times 100\% = \mathbf{29.80}\%$$

Problem Statement

A solid steel shaft of diameter $D = 75$ mm and length $L = 3.0$ m rotates at 450 RPM. Shear modulus $G = 80$ GPa. Maximum allowable shear stress is 55 MPa and maximum allowable twist is 1.5°. Determine the maximum permissible power (in kW and H.P.) that can be transmitted.

Step 1: Torque Based on Strength Constraint ($\tau_{\text{allow}} = 55$ MPa)

$$T_1 = \frac{\pi}{16}\tau_{\text{allow}} D^3 = \frac{\pi}{16} \times 55 \times 75^3 = 4,555,894 \text{ N} \cdot \text{mm} = 4555.89 \text{ N} \cdot \text{m}$$

Step 2: Torque Based on Stiffness Constraint ($\theta_{\text{allow}} = 1.5^\circ$)

Convert θ to radians: $\theta = 1.5 \times \frac{\pi}{180} = 0.02618$ rad. Length $L = 3000$ mm.

$$J = \frac{\pi D^4}{32} = \frac{\pi \times 75^4}{32} = 3,106,311 \text{ mm}^4$$

From torsion equation $\frac{T_2}{J} = \frac{G\theta}{L}$:

$$T_2 = \frac{G\theta J}{L} = \frac{80000 \times 0.02618 \times 3106311}{3000} = 2,168,435 \text{ N} \cdot \text{mm} = 2168.44 \text{ N} \cdot \text{m}$$

Step 3: Determine Permissible Power

Safe torque is governed by the smaller limit:
$T_{\text{perm}} = \min(T_1, T_2) = 2168.44\ \text{N} \cdot \text{m}$ (Stiffness governs!).

$$P_{\text{perm}} = \frac{2\pi N T_{\text{perm}}}{60000} = \frac{2\pi \times 450 \times 2168.44}{60000} = \mathbf{102.19}\ \text{kW}$$

$$\text{Power in Metric H.P.} = \frac{102.19}{0.7355} = \mathbf{138.94}\ \text{hp}$$

Table: Comprehensive Formula Cheat-Sheet for Shaft Torsion & Power Transmission

Design Quantity	Governing Equation	Key Units
Power Transmission	$P = \frac{2\pi NT}{60000}$	P in kW, T in N $\cdot$ m, N in RPM
Horsepower Conversion	$1\ \text{hp}_{\text{metric}} = 0.7355$ kW, $1\ \text{hp}_{\text{imperial}} = 0.7457$ kW	hp to kW
Peak Torque Factor	$T_{\text{max}} = k_t \cdot T_{\text{mean}}$	$k_t \geq 1.0$
Solid Shaft Strength	$T_{\text{max}} = \frac{\pi}{16}\tau_{\text{allow}} D^3$	D in mm, τ in N/mm^2
Hollow Shaft Strength	$T_{\text{max}} = \frac{\pi}{16}\tau_{\text{allow}} D_o^3(1 - k^4)$	$k = D_i/D_o$
Torsional Rigidity	$T = \frac{GJ\theta}{L}$	θ in rad, G in MPa

Key Engineering Takeaways

1. Always convert H.P. to Watts (W) or Kilowatts (kW) before calculating torque in $\mathrm{N \cdot m}$.
2. Always size transmission shafts based on peak torque T_{max}, not average torque T_{mean}.
3. For long transmission shafts, stiffness constraint ($\theta \leq \theta_{\text{allow}}$) frequently dictates shaft diameter over shear strength ($\tau \leq \tau_{\text{allow}}$).
4. Hollow shafts offer up to $\approx 30\%$ mass reduction for equivalent power transmission capacity.

Definition of Simple Lifting Machine

A simple lifting machine is a mechanical device designed to lift heavy loads (W) applied at one point by applying a comparatively smaller effort force (P) at another point, leveraging mechanical velocity conversion.

- **Fundamental Purpose**: Enables a single human operator or motor to lift heavy structural weights, equipment, or vehicles with reduced input force.
- **Law of Conservation of Energy**: Work Input = Work Output + Energy Losses due to Machine Friction.
- **Key Mechanical Performance Metrics**:
  - **Load (W)**: Total weight or resistance lifted by the machine (measured in N or kN).

- **Effort (P)**: Applied force required to lift the load (measured in N).
- **Mechanical Advantage (MA)**: Ratio of load lifted to effort applied, $\text{MA} = \frac{W}{P}$.
- **Velocity Ratio (VR)**: Ratio of distance moved by effort (y) to distance moved by load (x), $\text{VR} = \frac{y}{x}$.
- **Mechanical Efficiency (η)**: Percentage ratio of useful work output to work input, $\eta = \frac{\text{MA}}{\text{VR}} \times 100\%$.

Functional Classification of Machine Parts

Every simple lifting machine comprises four fundamental sub-assemblies working in kinematic sequence to transfer mechanical power.

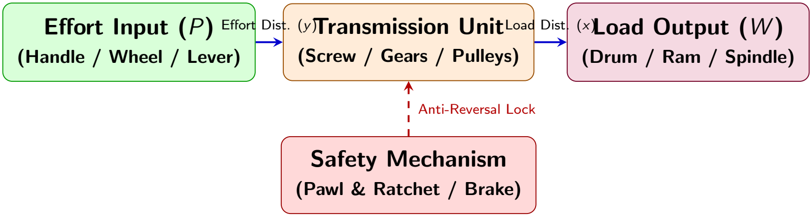
1. **Effort Input Mechanism**:
    - Accepts external manual or mechanical force P.
    - Examples: Tommy bar, effort wheel, crank handle, operating lever.
    - Primary kinematic variable: Effort displacement y.
2. **Power Transmission & Velocity Reduction Unit**:
    - Multiplies torque and reduces speed to provide mechanical advantage.
    - Examples: Square-threaded spindle, spur gears, worm gear drive, stepped axles.
3. **Load Supporting Mechanism**:

- Directly holds, guides, or raises the useful load W.
- Examples: Load drum, swivel cup, screw ram, hook, wire rope axle.
- Primary kinematic variable: Load displacement x.

4. **Structural Housing & Safety Controls**:
   - Maintains rigid alignment of shafts/threads and prevents catastrophic load falling.
   - Examples: Heavy cast iron frame, pawl and ratchet wheel, friction brake, guide sleeve.

System Kinematic Architecture

The flow of mechanical energy from effort input to load output involves internal friction losses across all bearing surfaces and meshing components.

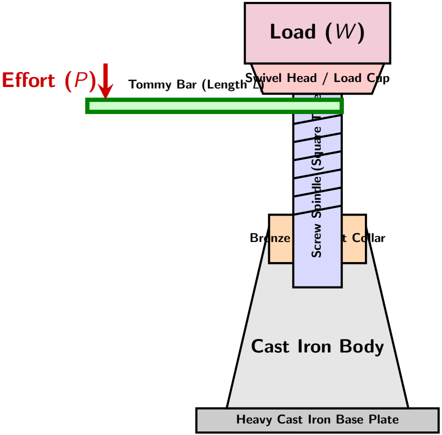


- **Friction Force in Machine**: $F = P - P_i = P - \frac{W}{VR}$, where P_i is ideal effort.

- **Frictional Lost Effort**: Additional effort required solely to overcome component friction.
- **Reversibility Condition**: Machine will reverse (overhaul) under load if $\eta > 50\%$. Self-locking occurs when $\eta < 50\%$.

Constructional Anatomy of Screw Jack

A simple screw jack relies on an inclined plane wrapped around a cylinder (square thread) to lift heavy vehicles and structural loads.



- **Tommy Bar**: Cylindrical handle fitted through the spindle head; provides effort radius L.

- **Screw Spindle**: Central steel rod cut with square threads of pitch p (converts rotation to axial lift).
- **Nut Collar**: Internally threaded bushing fixed inside the upper casing.
- **Swivel Head / Cup**: Free-swiveling top head that supports load W without rotating against it.
- **Velocity Ratio**: $\text{VR} = \frac{2\pi L}{p}$.

Working Principles of Differential Axles

Provides very high velocity ratios by utilizing two differential stepped axles mounted on a single effort shaft.

- **Effort Wheel (D)**: Large pulley of diameter D around which effort string is wound.
- **Larger Axle (d_1)**: Co-axial drum of diameter d_1 from which load rope winds up during lifting.
- **Smaller Axle (d_2)**: Co-axial drum of diameter d_2 from which load rope unwinds during lifting.
- **Movable Pulley**: Suspends load W within the bight of the continuous load rope loop.
- **Main Shaft & Support Bearings**: Holds effort wheel and stepped axles in rigid co-axial alignment.

- **Kinematic Displacement Analysis**:
  - In 1 turn of effort wheel, effort distance $y = \pi D$.
  - Net length of rope wound onto axles $= \pi d_1 - \pi d_2$.
  - Load lift distance $x = \frac{\pi(d_1 - d_2)}{2}$.
  - **Velocity Ratio**: $\text{VR} = \frac{y}{x} = \frac{2D}{d_1 - d_2}$.

High-Reduction Worm Gearing Assembly

Combines a helical threaded screw (worm) with a meshing gear wheel (worm wheel) to achieve massive force amplification in compact spaces.

- **Worm Shaft (Screw)**: Threaded horizontal steel shaft with single ($n = 1$) or double ($n = 2$) helical thread starts.
- **Worm Wheel (Gear)**: Toothed wheel with T teeth that meshes directly with the worm shaft.
- **Effort Drum / Pulley**: Mounted on the worm shaft (radius R or diameter D) to wrap effort rope.
- **Load Drum / Axle**: Keyed rigidly to the worm wheel shaft (radius r or diameter d) carrying the load rope.
- **Pawl and Ratchet Wheel**: Mounted on worm shaft to lock position and prevent back-driving under load.

- **Velocity Ratio Formula**:

$$\text{VR} = \frac{\text{Distance moved by Effort}}{\text{Distance moved by Load}} = \frac{D}{d} \cdot \frac{T}{n} = \frac{R}{r} \cdot \frac{T}{n}$$

- **Self-Locking Advantage**: High sliding friction between worm threads and gear teeth makes the unit inherently self-locking ($\eta < 50\%$).

Industrial Winch Crab Assemblies

Winch crabs use spur gear trains to lift building materials, heavy machinery, and structural components at construction sites.

Crank Handle / Lever Manual arm of radius R where operator applies effort P.

Pinion Gear (T_1) Small driving spur gear mounted on the effort crank shaft.

Main Spur Wheel (T_2) Large driven spur gear meshing with pinion T_1.

Intermediate Shaft & Pinion (T_3, T_4) Added in *Double Purchase Winch Crabs* to create a compound gear train ($T_1 \rightarrow T_2$ and $T_3 \rightarrow T_4$).

Load Cable Drum (d) Cylindrical drum of diameter d keyed to the main output gear shaft.

Side Frames & Bearings Cast iron side plates providing rigid structural support and shaft center distance alignment.

- **Single Purchase Winch Crab VR**: $\mathrm{VR} = \frac{2R}{d} \cdot \frac{T_2}{T_1}$.
- **Double Purchase Winch Crab VR**: $\mathrm{VR} = \frac{2R}{d} \cdot \left(\frac{T_2 \cdot T_4}{T_1 \cdot T_3}\right)$.

Comparative Engineering Parameters

Quantitative comparison of kinematic expressions, design dimensions, and mechanical efficiency ranges across standard machines.

Table: Kinematic Formulas and Performance Metrics of Simple Lifting Machines

Machine Type	**Velocity Ratio (VR)**	**Key Design Parameters**	**Typ. Efficiency (η)**
Simple Screw Jack	$\frac{2\pi L}{p}$	$L = 500$ mm, Pitch $p = 10$ mm	$35\% - 50\%$
Differential Wheel & Axle	$\frac{2D}{d_1 - d_2}$	$D = 400$ mm, $d_1 = 150$ mm, $d_2 = 100$ mm	$60\% - 75\%$
Worm & Worm Wheel	$\frac{D}{d} \cdot \frac{T}{n}$	$T = 40$ teeth, $n = 1$ start, $D/d = 2$	$40\% - 60\%$
Single Purchase Crab	$\frac{2R}{d} \cdot \frac{T_2}{T_1}$	$R = 400$ mm, $d = 200$ mm, $\frac{T_2}{T_1} = 4$	$65\% - 80\%$
Double Purchase Crab	$\frac{2R}{d} \cdot \frac{T_2 \cdot T_4}{T_1 \cdot T_3}$	$R = 400$ mm, $d = 200$ mm, Gear Ratio $= 16$	$60\% - 75\%$

- Higher velocity ratio allows lifting massive loads with minimal human effort.
- Machines with efficiency below 50% are self-locking and prevent sudden reverse dropping of load.

Material Selection and Stress Criteria

Component design requires matching engineering materials to specific mechanical stress states during operation.

Table: Engineering Material Specifications and Stress Types for Machine Components

Component	**Recommended Material**	**Primary Stress Mode**	**Design Considerations**
Screw Spindle	High Carbon Steel (C45 / EN8)	Compression & Torsion	Square threads (high efficiency)
Nut Body	Phosphor Bronze / Cast Iron	Shear & Bearing Stress	Low friction coefficient ($\mu \approx 0.12$)
Machine Frame/Base	Grey Cast Iron (FG 200 / FG 250)	Heavy Compression	High vibration damping capacity
Gears & Pinions	Alloy Steel (20CrMnTi / Cast Steel)	Bending & Surface Wear	Involute tooth profile (20° pressure angle)
Tommy Bar / Handle	Mild Steel (Fe 410 / C20)	Pure Bending Stress	High ductility to prevent brittle failure
Load Cable / Rope	Galvanized Steel Wire Rope	Pure Tension	High factor of safety (FOS ≥ 5)

- Maintenance requirement: Regular lubrication on threads, gears, and pillow block bearings to minimize friction loss F.

- **Definition**: A simple lifting machine is a mechanical device designed to lift heavy loads (W) at a specific point by applying a comparatively smaller effort force (P) at another point.
- **Mechanical Advantage (MA)**: The ratio of the load lifted (W) to the effort applied (P).

$$\text{MA} = \frac{\text{Load } (W)}{\text{Effort } (P)} = \frac{W}{P} \tag{58}$$

- **Velocity Ratio (VR)**: The ratio of the distance moved by the effort (y) to the distance moved by the load (x) in the same time interval.

$$\text{VR} = \frac{\text{Distance moved by Effort } (y)}{\text{Distance moved by Load } (x)} = \frac{y}{x} \tag{59}$$

- **Efficiency (η)**: The ratio of output work done by the machine to input work supplied to the machine.

$$\eta = \frac{\text{Output Work}}{\text{Input Work}} = \frac{W \cdot x}{P \cdot y} = \frac{\text{MA}}{\text{VR}} \quad \Longrightarrow \quad \eta(\%) = \frac{\text{MA}}{\text{VR}} \times 100 \tag{60}$$

- **Ideal Machine vs. Real Machine**:
  - **Ideal Machine**: No frictional losses ($\eta = 100\% \implies \text{MA} = \text{VR}$). Ideal Effort $P_i = \frac{W}{\text{VR}}$.

- **Effort Lost in Friction (P_f)**: $P_f = P - P_i = P - \frac{W}{\mathrm{VR}}$.
- **Frictional Load Loss (W_f)**: $W_f = W_i - W = (P \cdot \mathrm{VR}) - W$.

- **Law of Machine**: An empirical linear equation defining the relationship between the effort P required to lift a load W for a given simple lifting machine.

$$P = m \cdot W + C \tag{61}$$

where:

  - m = slope of the line $(\Delta P / \Delta W)$, representing the fractional effort increase per unit load.
  - C = effort intercept at $W = 0$, representing the effort required to overcome initial machine friction at zero load.

- **Maximum Mechanical Advantage ($\mathbf{MA_{max}}$)**:

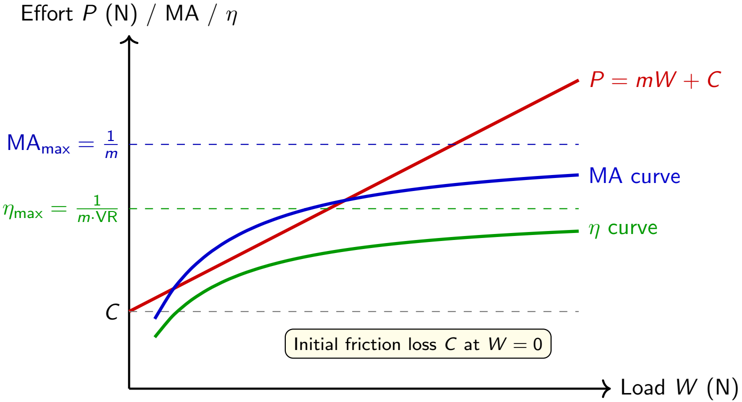
$$\text{MA} = \frac{W}{P} = \frac{W}{mW + C} = \frac{1}{m + \frac{C}{W}} \implies \text{MA}_{\text{max}} = \lim_{W \to \infty} \text{MA} = \frac{1}{m} \tag{62}$$

- **Maximum Efficiency (η_{max})**:

$$\eta_{\text{max}} = \frac{\text{MA}_{\text{max}}}{\text{VR}} = \frac{1}{m \cdot \text{VR}} \quad \Longrightarrow \quad \eta_{\text{max}}(\%) = \frac{1}{m \cdot \text{VR}} \times 100 \tag{63}$$

- **Engineering Significance**: As load W increases, MA and η asymptote toward their maximum theoretical limits, while friction loss increases linearly with load.

- **Graphical Representation**: The effort-load characteristic is a straight line with slope m and positive intercept C. The Mechanical Advantage and Efficiency curves follow rectangular hyperbolic paths.



- **Reversible Machine**:
  - A machine capable of working in the reverse direction when effort P is completely released, causing the load W to descend automatically.
  - **Condition for Reversibility**: Output work must be greater than frictional work loss ($W_{\text{out}} > W_f$).

$$\text{Efficiency } \eta = \frac{\text{Output Work}}{\text{Input Work}} > 50\% \tag{64}$$

- **Self-Locking Machine (Non-Reversible)**:
  - A machine that does not move in the reverse direction when effort is removed because friction holds the load in place.

- **Condition for Self-Locking**: Frictional work loss exceeds or equals output work ($W_f \geq W_{\text{out}}$).

$$\text{Efficiency } \eta = \frac{\text{Output Work}}{\text{Input Work}} \leq 50\% \tag{65}$$

- **Industrial Applications of Self-Locking**:
  - **Screw Jacks**: Used for automotive lifting so that the elevated car does not slip back down when handle effort is removed.
  - **Worm & Worm Wheel Hoists**: Used in overhead cranes to ensure load holding safety.

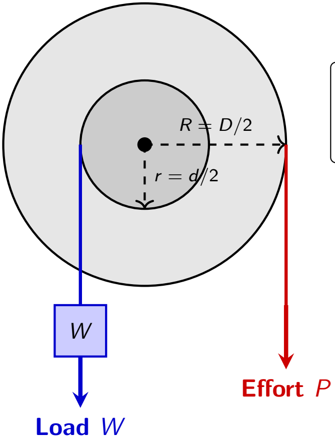
- **Kinematic Analysis**: The Velocity Ratio (VR) depends strictly on the geometric dimensions of the machine mechanism.

Machine Type	Key Dimensions	Velocity Ratio (VR) Formula
Simple Wheel & Axle	D: Effort Wheel Dia. d: Load Axle Dia.	$\text{VR} = \frac{D}{d}$
Differential Wheel & Axle	D: Effort Wheel Dia. d_1, d_2: Axle Diameters	$\text{VR} = \frac{2D}{d_1 - d_2}$
Weston's Differential Pulley	D: Larger Pulley Dia. d: Smaller Pulley Dia.	$\text{VR} = \frac{2D}{D-d} = \frac{2T_1}{T_1 - T_2}$
Worm & Worm Wheel	R: Effort Arm Radius r: Load Drum Radius T: Teeth on Wheel n: Worm Threads	$\text{VR} = \frac{R}{r} \cdot \frac{T}{n}$

- **Gearing & Screw Mechanisms**: Higher velocity ratios are achieved by combining multi-stage gear trains or fine pitch threads.

Machine Type	Key Dimensions	Velocity Ratio (VR) Formula
Single Purchase Winch Crab	L: Handle Radius r: Drum Radius T_1: Pinion Teeth T_2: Spur Gear Teeth	$\mathrm{VR} = \frac{L}{r} \cdot \left(\frac{T_2}{T_1}\right)$
Double Purchase Winch Crab	L: Handle Radius r: Drum Radius T_1, T_3: Pinion Teeth T_2, T_4: Spur Teeth	$\mathrm{VR} = \frac{L}{r} \cdot \left(\frac{T_2 \cdot T_4}{T_1 \cdot T_3}\right)$
Simple Screw Jack	L: Tommy Bar Length p: Screw Pitch	$\mathrm{VR} = \frac{2\pi L}{p}$
Differential Screw Jack	L: Tommy Bar Length p_1, p_2: Screw Pitches	$\mathrm{VR} = \frac{2\pi L}{p_1 - p_2}$

- **Kinematic Mechanism**: Effort P is applied via a string wound over effort wheel of diameter D. Load W is lifted by string wound around axle of diameter d.



Per 1 Revolution:
Effort Displacement $y = \pi D$
Load Displacement $x = \pi d$
$\text{VR} = \frac{y}{x} = \frac{\pi D}{\pi d} = \frac{D}{d}$

- **Laboratory Test Dataset**: Performance metrics recorded for a Simple Screw Jack having fixed Velocity Ratio VR $= 314.16$.

Test No.	Load W (N)	Effort P (N)	MA $= \frac{W}{P}$	VR	Efficiency η (%)	Friction P_f (N)
1	500	22.0	22.73	314.16	7.23	20.41
2	1000	34.0	29.41	314.16	9.36	30.82
3	1500	46.0	32.61	314.16	10.38	41.23
4	2000	58.0	34.48	314.16	10.98	51.63
5	2500	70.0	35.71	314.16	11.37	62.04
6	3000	82.0	36.59	314.16	11.65	72.45

- **Regression Analysis**:
    - Linear Law derived: $P = 0.024W + 10.0\,\text{N}$.
    - Slope $m = 0.024$, Intercept $C = 10.0\,\text{N}$.
    - Theoretical Maximum $\text{MA}_{\text{max}} = \frac{1}{0.024} = 41.67$.
    - Theoretical Maximum Efficiency $\eta_{\text{max}} = \frac{41.67}{314.16} \times 100 = 13.26\%$.
    - Since $\eta < 50\%$, the machine is strictly **self-locking**.

- **Problem Statement**: A simple screw jack has a thread pitch $p = 10\,\text{mm}$ and a tommy bar length $L = 500\,\text{mm}$. It is used to lift a load $W = 12\,\text{kN}$ by applying an effort $P = 150\,\text{N}$. Calculate: (a) Velocity Ratio (VR), (b) Mechanical Advantage (MA), (c) Efficiency (η), (d) Effort lost in friction (P_f), and (e) Check if machine is self-locking.
- **Step 1: Calculate Velocity Ratio (VR)**:

$$\text{VR} = \frac{2\pi L}{p} = \frac{2\pi \times 500\,\text{mm}}{10\,\text{mm}} = 314.16 \tag{66}$$

- **Step 2: Calculate Mechanical Advantage (MA)**:

$$\text{MA} = \frac{W}{P} = \frac{12000\,\text{N}}{150\,\text{N}} = 80.0 \tag{67}$$

- **Step 3: Calculate Efficiency (η)**:

$$\eta = \frac{\text{MA}}{\text{VR}} \times 100 = \frac{80.0}{314.16} \times 100 = 25.46\% \tag{68}$$

- **Step 4: Calculate Effort Lost in Friction (P_f)**:

$$P_i = \frac{W}{\text{VR}} = \frac{12000}{314.16} = 38.20\,\text{N} \quad \Longrightarrow \quad P_f = P - P_i = 150 - 38.20 = 111.80\,\text{N} \tag{69}$$

- **Step 5: Reversibility Check**:

$$\eta = 25.46\% < 50\% \quad \Longrightarrow \quad \textbf{Machine is Self-Locking.} \tag{70}$$

- **Problem Statement**: In a lifting machine with VR = 40, an effort of 60 N lifts a load of 1000 N, and an effort of 140 N lifts a load of 3000 N. Determine: (a) Law of the machine, (b) Effort required for $W = 4500\,\text{N}$, (c) Maximum Mechanical Advantage, and (d) Maximum Efficiency.
- **Step 1: Set Up Simultaneous Equations ($P = mW + C$)**:

$$60 = 1000m + C \quad \text{— (Equation 1)} \tag{71}$$
$$140 = 3000m + C \quad \text{— (Equation 2)} \tag{72}$$

- **Step 2: Solve for m and C**: Subtracting Eq. 1 from Eq. 2:

$$80 = 2000m \implies m = \frac{80}{2000} = 0.04 \tag{73}$$

Substitute $m = 0.04$ into Eq. 1:

$$C = 60 - 1000(0.04) = 60 - 40 = 20\,\text{N} \tag{74}$$

Law of Machine: $\mathbf{P = 0.04W + 20}$

- **Step 3: Effort for $W = 4500\,\mathbf{N}$**:

$$P_{4500} = 0.04(4500) + 20 = 180 + 20 = 200\,\text{N} \tag{75}$$

- **Step 4: Maximum MA and Maximum Efficiency**:

$$MA_{max} = \frac{1}{m} = \frac{1}{0.04} = 25.0 \tag{76}$$

$$\eta_{max} = \frac{MA_{max}}{VR} \times 100 = \frac{25}{40} \times 100 = 62.5\% \tag{77}$$

Learning Objectives

- Define **Reversible** and **Irreversible (Self-locking)** simple lifting machines.
- Derive the mathematical threshold condition for reversibility based on efficiency ($\eta > 50\%$).
- Analyze energy flow, work done against load, and friction losses in lifting mechanisms.
- Evaluate power screw parameters: Helix Angle (α) versus Angle of Friction (ϕ).
- Compare machine characteristics using real engineering data and practical design criteria.

Engineering Significance of Reversibility

- **Safety in Hoisting:** Prevents sudden dropping of heavy suspended loads when effort is released (e.g., car jacks, crane winches).
- **Energy Efficiency:** High efficiency ($\eta > 50\%$) is desired for continuous power transmission, whereas self-locking ($\eta \leq 50\%$) is mandatory for load retention.

Definitions

- **Reversible Machine:** A machine in which the load W moves downward and drives the machine in the reverse direction when the effort P is completely removed. This phenomenon is known as **overhauling**.
- **Irreversible (Self-locking) Machine:** A machine in which the load W remains in its lifted position and does not move downward when effort P is removed, owing to internal friction resisting reverse motion.

Work & Energy Breakdown during Lifting

Consider a machine lifting load W through distance x by applying effort P moving through distance y:

- **Input Work (W_{input}):** Energy supplied by effort $= P \times y$
- **Useful Output Work (W_{output}):** Work done on load $= W \times x$
- **Friction Work ($W_{friction}$):** Energy dissipated by internal friction $= P \cdot y - W \cdot x$

Condition for Reverse Motion

When effort P is removed, the load W attempts to move back down distance x, yielding available work equal to $W \times x$. To drive the machine in reverse, this available work must be greater than the friction work ($W_{friction}$) of the machine.

Step-by-Step Proof

$$
\begin{aligned}
W_{output} &> W_{friction} \\
W \cdot x &> (P \cdot y - W \cdot x) \\
W \cdot x + W \cdot x &> P \cdot y \\
2 \cdot (W \cdot x) &> P \cdot y \\
\frac{W \cdot x}{P \cdot y} &> \frac{1}{2} = 0.5
\end{aligned}
$$

Since Mechanical Efficiency is $\eta = \frac{W \cdot x}{P \cdot y}$:

$$\eta > 0.5 \quad \text{or} \quad \eta > 50\%$$

Self-Locking Condition

If $W_{output} \leq W_{friction}$, the machine cannot drive itself in reverse:

$$\eta \leq 0.5 \quad \text{or} \quad \eta \leq 50\%$$

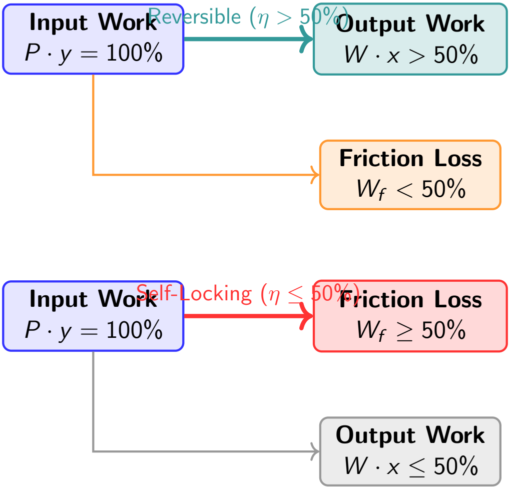

Input Work
$P \cdot y = 100\%$
Reversible ($\eta > 50\%$)
Output Work
$W \cdot x > 50\%$
Friction Loss
$W_f < 50\%$
Input Work
$P \cdot y = 100\%$
Self-Locking ($\eta \leq 50\%$)
Friction Loss
$W_f \geq 50\%$
Output Work
$W \cdot x \leq 50\%$

Table: Engineering Comparison Matrix of Lifting Machines

Parameter	**Reversible Machine**	**Irreversible (Self-Locking)**
Efficiency Threshold	$\eta > 50\%$	$\eta \leq 50\%$
Useful Output Work	$W_{out} > W_{friction}$	$W_{out} \leq W_{friction}$
Behavior on Effort Removal	Load moves downward automatically (Overhauling)	Load remains sustained in position
Safety Risk	High risk of uncontrolled load dropping	Inherently safe against back-driving
Safety Device Requirement	Requires external brake, pawl, or friction clutch	No auxiliary brake needed for holding load
Screw Geometry (α vs ϕ)	Helix angle $\alpha >$ Friction angle ϕ	Helix angle $\alpha \leq$ Friction angle ϕ
Typical Applications	Differential pulley blocks, crane hoists, gear trains	Screw jacks, bench vices, worm gear winches

Screw Jack Mechanics

For a square-threaded power screw with helix angle α and friction angle $\phi = \tan^{-1}(\mu)$:

- **Effort to Raise Load:** $P = W \tan(\alpha + \phi)$
- **Effort to Lower Load:** $P' = W \tan(\phi - \alpha)$
- **Efficiency (η):**

$$\eta = \frac{W \tan \alpha}{W \tan(\alpha + \phi)} = \frac{\tan \alpha}{\tan(\alpha + \phi)}$$

Conditions for Reversibility in Screws

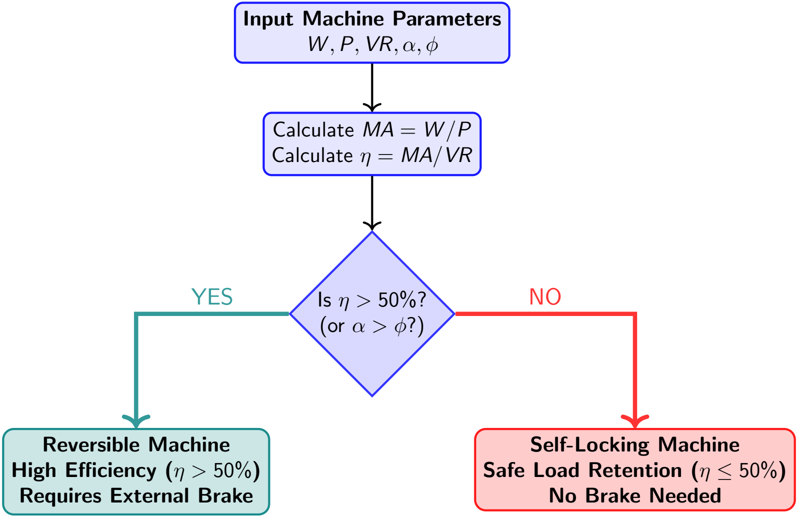
- **If $\alpha > \phi$:** Effort to lower load $P' > 0$. When effort is removed, P' becomes negative, meaning the load descends spontaneously ($\eta > 50\%$).
- **If $\alpha \leq \phi$:** Effort to lower load $P' \leq 0$. Friction holds the screw in place without effort ($\eta \leq 50\%$).

Table: Performance Data of Standard Engineering Lifting Machines

Machine Type	**Load W (kN)**	**Effort P (N)**	**VR**	**MA**	**Efficiency η**	**Classification**
Simple Screw Jack	10.0	250.0	104.7	40.00	38.20%	Irreversible (Self-Locking)
Differential Pulley	5.0	120.0	60.0	41.67	69.45%	Reversible (Overhauling)
Worm & Worm Wheel	15.0	450.0	100.0	33.33	33.33%	Irreversible (Self-Locking)
Single Purchase Winch	2.0	50.0	65.0	40.00	61.54%	Reversible (Overhauling)
Bench Vice Screw	8.0	300.0	80.0	26.67	33.33%	Irreversible (Self-Locking)

Data Analysis

- Machines with high Velocity Ratio ($VR > 80$) often experience high internal friction, keeping efficiency below 50% (Irreversible).
- Pulley systems and simple winches achieve efficiencies up to 70%, necessitating ratchet-and-pawl mechanisms for safety.


Input Machine Parameters
W, P, VR, α, ϕ
Calculate $MA = W/P$
Calculate $\eta = MA/VR$
Is $\eta > 50\%$?
(or $\alpha > \phi$?)
YES
NO
Reversible Machine
High Efficiency ($\eta > 50\%$)
Requires External Brake
Self-Locking Machine
Safe Load Retention ($\eta \leq 50\%$)
No Brake Needed

Selection Criteria in Mechanical Design

- **Use Self-Locking Machines ($\eta \leq 50\%$):**
  - Automotive Screw Jacks (prevents vehicle from dropping during wheel replacement).
  - Machine Tool Vices and Clamps (maintains clamping force without continuous input).
  - Elevator Worm Gear Drives (failsafe hold in case of motor power outage).
- **Use Reversible Machines ($\eta > 50\%$):**
  - Material Handling Cranes and High-Speed Hoists (minimizes power consumption).
  - Simple Lever and Pulley Systems (reduces operator effort for frequent lifting).

Key Takeaways

- Reversibility condition: $\eta > 50\%$ ($W_{output} > W_{friction}$).
- Irreversibility / Self-locking condition: $\eta \leq 50\%$ ($W_{output} \leq W_{friction}$).
- Screw mechanisms are self-locking when helix angle $\alpha \leq$ friction angle ϕ.

- **Definition of Simple Lifting Machine:** A mechanical device designed to lift heavy loads (W) by applying a relatively smaller effort (P) through a mechanical transmission system.
- **Concept of Eco-Friendly Lifting Systems:** Modern lifting machinery engineered to maximize energy efficiency (η), minimize carbon emissions, eliminate toxic hydraulic fluid leakage, and integrate renewable power or human actuation.
- **Fundamental Performance Equations:**
  - Mechanical Advantage (MA): $MA = \frac{W}{P}$
  - Velocity Ratio (VR): $VR = \frac{y}{x}$ (where y is effort displacement and x is load displacement)
  - Mechanical Efficiency (η): $\eta = \frac{MA}{VR} \times 100\%$
- **Key Environmental Selection Drivers:**
  - Minimization of frictional dissipation and heat waste ($F_p = P - \frac{W}{VR}$).

- Usage of biodegradable fluids (HEES synthetic esters / HETG vegetable oils).
- Operational safety, self-locking features, and long lifecycle endurance.

- **Law of Lifting Machine:** Empirical linear relation between applied effort P and lifted load W:

$$P = m \cdot W + C$$

where m is the friction coefficient constant and C represents zero-load parasitic effort.

- **Ideal Effort vs. Friction Effort:**
  - Ideal Effort (P_i): Effort required without friction, $P_i = \frac{W}{VR}$
  - Effort Lost in Friction (F_p): $F_p = P - P_i = P - \frac{W}{VR}$
  - Load Lost in Friction (F_w): $F_w = W - (P \times VR)$

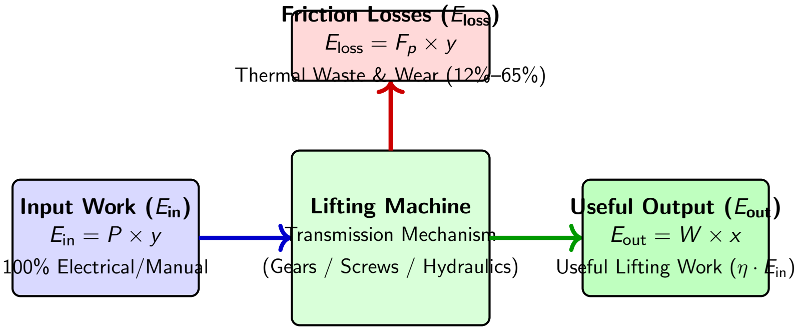
- **Maximum Asymptotic Mechanical Efficiency ($\eta_{\max}$):**

$$\eta_{\max} = \frac{1}{m \cdot VR} \times 100\%$$

- **Ecological Significance of Energy Loss:**
  - Low efficiency ($\eta < 50\%$) doubles energy consumption and carbon emissions.

- Parasitic friction causes component wear, generating metallic particulate waste.
- High-efficiency machines ($\eta > 80\%$) drastically lower electrical grid load.

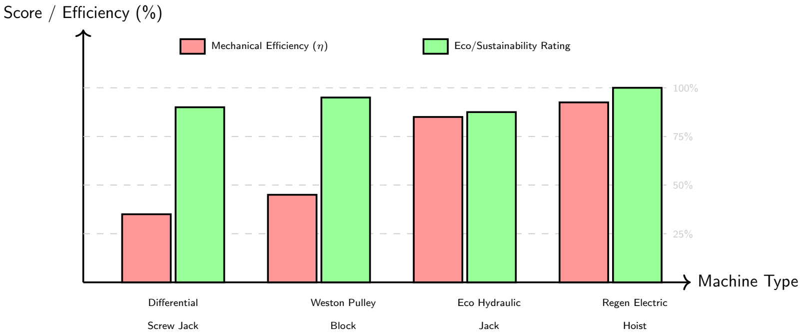
Machine Type	**Velocity Ratio (VR)**	**Typical Efficiency (η)**	**Energy Source**	**Eco-Rating**
Differential Screw Jack	$\frac{2\pi R}{p_1 - p_2} \approx 60 - 150$	$28\% - 42\%$	Manual / Mechanical	Moderate
Weston's Differential Pulley Block	$\frac{2D}{D-d} \approx 20 - 60$	$35\% - 48\%$	Manual Chain	High (Zero Emission)
Worm & Worm Wheel Hoist	$\frac{T \cdot R}{r} \approx 30 - 120$	$45\% - 70\%$	Manual / Electric	Good
Eco-Hydraulic Bottle Jack	$\left(\frac{D}{d}\right)^2 \frac{L}{l} \approx 100 - 400$	$75\% - 88\%$	Bio-Fluid / Manual	Superior
Regenerative Electric Chain Hoist	Geared Ratio $\approx 40 - 200$	$85\% - 94\%$	Grid / Solar + Regen	Excellent



- **Energy Audit Principle:** Total input work equals useful work plus frictional losses ($E_{\text{in}} = E_{\text{out}} + E_{\text{loss}}$).
- **Eco-Optimization Target:** Reduce E_{loss} via low-friction synthetic ester bio-lubricants, roller needle bearings, and optimized thread pitch geometries.

- **Engineering Selection Criteria:**
  1. **Operational Duty Cycle:** Frequency of lifting operations per hour/day.
  2. **Power Availability:** On-grid electrical supply vs. off-grid manual/solar operation.
  3. **Environmental Sensitivity:** Zero-spill requirements in pristine/agricultural zones.
  4. **Self-Locking Safety Requirement:** Necessity of $\eta < 50\%$ to prevent accidental load descent without power brake.
- **Purpose-Driven Recommendation Rules:**
  - **Off-Grid Heavy Emergency Lifting ($W > 50$ kN):** Eco-Hydraulic Bottle Jack with bio-based hydraulic oil.
  - **High-Frequency Manufacturing Assembly Line:** Regenerative Electric Chain Hoist with energy feedback braking.
  - **Precision Maintenance & Workshop Support:** Weston's Differential Pulley Block or Differential Screw Jack.

6. Eco-Efficiency & Sustainability Evaluation Chart I



- **Operational Requirement:** Lift a vehicle of mass $M = 2500$ kg ($W = 24.53$ kN) through $h = 1.8$ m in an eco-certified service center.
- **Option Evaluation & Numerical Comparison:**
  - **Option A (Mechanical Screw Lift):** $\eta = 40\%$, $VR = 100$.
    - Useful Work: $W_{\text{out}} = W \times h = 24.53 \text{ kN} \times 1.8 \text{ m} = 44.15 \text{ kJ}$
    - Required Energy Input: $E_{\text{in, A}} = \frac{44.15}{0.40} = 110.38 \text{ kJ}$
  - **Option B (Eco-Hydraulic Bio-Fluid Lift):** $\eta = 88\%$, $VR = 250$.
    - Required Energy Input: $E_{\text{in, B}} = \frac{44.15}{0.88} = 50.17 \text{ kJ}$
- **Engineering Justification:** Option B reduces energy consumption by **54.5%** (60.21 kJ saved per lift cycle) while utilizing biodegradable synthetic ester oil, rendering it the optimal eco-friendly choice.

- **Operational Requirement:** Raise a 750 kg submersible pump ($W = 7.36$ kN) at an off-grid agricultural solar station with single-operator manual actuation ($P_{\text{human}} \leq 250$ N).
- **Candidate Evaluation:**
  - **Simple Winch Drum ($VR = 30$, $\eta = 60\%$):**

    $$P = \frac{W}{VR \cdot \eta} = \frac{7360 \text{ N}}{30 \times 0.60} = 408.9 \text{ N} \quad \text{(Exceeds human force limit of 250 N)}$$
  - **Weston's Differential Pulley Block ($VR = 50$, $\eta = 42\%$):**

    $$P = \frac{7360 \text{ N}}{50 \times 0.42} = 350.5 \text{ N} \quad \text{(Still excessive for continuous operation)}$$
  - **High-VR Geared Worm Hoist ($VR = 120$, $\eta = 55\%$):**

    $$P = \frac{7360 \text{ N}}{120 \times 0.55} = 111.5 \text{ N} \quad \text{(Safely within human capacity!)}$$
- **Engineering Justification:** High-VR Geared Worm Hoist selected because it enables zero-carbon manual operation well within human ergonomic limits while providing self-locking safety.

- **Comprehensive Selection Checklist for Eco-Lifting Machines:**
  1. **Match VR to Input Source:** Select VR such that effort $P \leq P_{\text{allowable}}$ (human ergonomic limit or motor power rating).
  2. **Maximize Efficiency for High Duty Cycles:** Choose $\eta \geq 80\%$ (eco-hydraulic or regenerative electric hoists) for continuous operations to minimize carbon emissions.
  3. **Enforce Environmental Fluid Standards:** Mandate ISO 15380 HEES/HETG non-toxic biodegradable fluids for hydraulic machinery.
  4. **Evaluate Self-Locking Safety Trade-Offs:** Balance safety requirements ($\eta < 50\%$) against energy loss penalties.
- **Conclusion:** Eco-friendly lifting machine selection requires aligning velocity ratio, mechanical efficiency, environmental fluid safety, and energy recovery mechanisms with the specific operational purpose.