Theory of Machines and Mechanisms (DI03000141)

GTU Course Presentation

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Course Overview: Theory of Machines and Mechanisms I

GTU Course: DI03000141

- **Course Title:** Theory of Machines and Mechanisms
- **Unit 1:** Motions & Mechanisms
- **Lecture 1:** 1.1 Theory of Machines – Fundamentals, Links, Pairs, and Kinematic Chains

Course Overview: Theory of Machines and Mechanisms II

Lecture Objectives

- Understand the fundamental definitions and scope of Theory of Machines.
- Identify the core subdivisions: Kinematics, Dynamics, Statics, and Kinetics.
- Classify kinematic links, joints, and pairs based on motion and contact.
- Apply Grübler's criterion for evaluating Degrees of Freedom (DoF).
- Analyze planar four-bar mechanisms, Grashof's law, and kinematic inversions.

Definition & Scope of Theory of Machines I

Fundamental Definition

Theory of Machines is that branch of Engineering Science which deals with the study of relative motion between the various parts of a machine, and the forces which act on them.

Definition & Scope of Theory of Machines II

Key Scope and Engineering Relevance

- **Analytical Foundation:** Provides the mathematical basis for designing mechanisms, linkage assemblies, gears, cams, and robotic manipulators.
- **Kinematic & Dynamic Analysis:** Enables accurate computation of displacements, velocities, accelerations, and dynamic forces in mechanical systems.
- **Design Synthesis:** Bridges the gap between geometric motion synthesis and structural/dynamic integrity under actual operating loads.
- **Efficiency & Precision:** Minimizes wear, vibration, and energy losses through optimized mechanism design.

Subdivisions of Theory of Machines I

Core Subdivisions

The study of Theory of Machines is systematically divided into four major branches:

Kinematics of Machines: Study of relative motion (displacement, velocity, and acceleration) of machine parts without considering the forces causing the motion.

Dynamics of Machines: Study of forces acting on machine parts and their effect on the resulting motion.

- **Statics:** Deals with forces acting on machine parts when they are assumed to be stationary or when inertia forces are neglected.
- **Kinetics:** Deals with inertia forces arising from the combined effect of mass and acceleration of moving parts.

Kinematic Link or Element I

Definition of Kinematic Link

A **kinematic link** (or element) is a resistant body or a group of resistant bodies that forms a part of a machine and has relative motion with respect to another part.

Essential Characteristics

A body is called a resistant body if it can transmit the required forces or motion with negligible deformation.

Kinematic Link or Element II

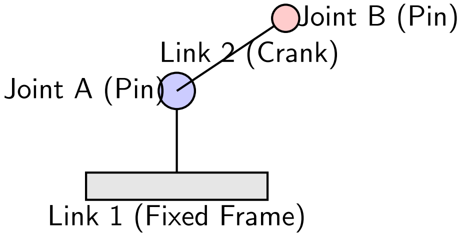
Classification of Links

- **Rigid Link:** Does not undergo detectable deformation while transmitting motion (e.g., crank, connecting rod, piston).
- **Flexible Link:** Deforms in a controlled, flexible manner to transmit tensile forces (e.g., belts, ropes, chains, flexible shafts).
- **Fluid Link:** Transmits power or motion through fluid pressure and hydrostatic action (e.g., hydraulic oil in jacks, fluid clutches, hydraulic brakes).

Kinematic Joint (Revolute Pair) I

Kinematic Joint Concept

A joint or pair is formed by the connection of two links, allowing defined relative motion between them.



- **Revolute (Pin) Joint:** Restricts relative motion to pure rotation around a single central axis.
- **Lower Pair Nature:** Maintains surface contact across the joint, maximizing load distribution and reducing wear.

Categories of Kinematic Pairs

Kinematic pairs are classified based on relative motion, type of contact, and physical constraint:

- **By Nature of Relative Motion:**
  - *Sliding Pair (Prismatic):* Relative motion is pure translation (e.g., piston in cylinder).
  - *Turning Pair (Revolute):* Relative motion is pure rotation (e.g., shaft in journal bearing).
  - *Rolling Pair:* Relative motion is pure rolling without slipping (e.g., ball bearings).
  - *Screw Pair (Helical):* Relative motion combines rotation and translation via threads (e.g., lead screw of lathe).
  - *Spherical Pair:* Relative motion is 3D rotation about a point (e.g., ball and socket joint).
- **By Nature of Contact:**

- *Lower Pairs:* Surface/area contact between links.
- *Higher Pairs:* Line or point contact between links (e.g., cam-follower, meshing gears).

- **By Nature of Constraint:**
  - *Closed Pairs:* Held together geometrically/mechanically.
  - *Unclosed Pairs:* Held together by force (gravity or spring).

Summary of Kinematic Pairs I

Table: Classification and Examples of Kinematic Pairs

Pair Category & **Contact Type** & **DoF** & **Practical Example**
Sliding Pair & Lower (Surface) & 1 & Piston & Cylinder
Turning Pair & Lower (Surface) & 1 & Shaft in Journal Bearing
Rolling Pair & Higher (Line/Point) & 1 & Ball/Roller Bearings
Screw Pair & Lower (Surface) & 1 & Lead Screw & Nut
Spherical Pair & Lower (Surface) & 3 & Ball & Socket Joint
Cam Pair & Higher (Line/Point) & 2 & Cam & Knife-edge Follower

Degrees of Freedom & Grübler's Criterion I

Degrees of Freedom (DoF)

Degrees of Freedom (DoF) is the number of independent coordinates needed to uniquely define the position and configuration of a mechanism.

Grübler's Formula for Planar Mechanisms

For a planar mechanism with n links, j lower pairs (1 DoF each), and h higher pairs (2 DoF each):

$$F = 3(n-1) - 2j - h$$

Degrees of Freedom & Grübler's Criterion II

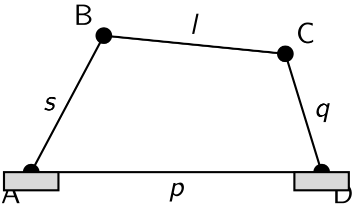
Interpretation of Mobility (F)

- $F = 0$**:** Statically determinate structure/frame (no relative motion).
- $F < 0$**:** Super-statical / redundant structure (pre-stressed, locked linkage).
- $F \geq 1$**:** Constrained mechanism (movable with F independent inputs).

The Planar Four-Bar Mechanism I

Four-Bar Linkage Overview

The planar four-bar mechanism is the simplest closed-loop kinematic chain consisting of four links connected by four revolute joints.



The Planar Four-Bar Mechanism II

Grashof's Criterion

For a four-bar linkage, if $s + l \leq p + q$ (where s is the shortest link, l is the longest, and p, q are intermediate links), at least one link can make a complete 360° rotation relative to the frame.

Kinematic Chains vs. Mechanisms I

Definitions

- **Kinematic Chain:** An assembly of links connected by kinematic pairs such that relative motion between links is completely constrained or successfully constrained.
- **Mechanism:** Obtained by fixing any one link of a kinematic chain to form a grounded frame.

Kinematic Chains vs. Mechanisms II

Kinematic Inversion

- **Definition:** The process of obtaining different mechanisms by grounding different links of the same kinematic chain.
- **Invariance Property:** The relative motion between individual links remains unchanged during inversion, but absolute motion relative to the frame changes.
- **Number of Inversions:** A kinematic chain with n links can yield up to n distinct inversions.

Single Slider Crank Chain Inversions

- **First Inversion (Cylinder fixed):** Reciprocating engine, air compressor.
- **Second Inversion (Crank fixed):** Whitworth quick return mechanism, rotary engine.
- **Third Inversion (Connecting rod fixed):** Crank and slotted lever mechanism, oscillating cylinder engine.

Kinematic Synthesis Objectives

- **Function Generation:** Coordinating input link position/velocity with output link position/velocity (e.g., function generator linkages).
- **Path Generation:** Designing a mechanism so that a coupler point traces a precise spatial curve (e.g., straight-line linkages like Watt's mechanism).
- **Motion Generation:** Guiding an entire rigid body through a sequence of prescribed orientations and positions (e.g., pick-and-place industrial robots).

Lecture Overview: Theory of Machines

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit:** Motions & Mechanisms
- **Topic:** 1.1 Theory of Machines

Learning Objectives

- Understand the definition, scope, and engineering significance of Theory of Machines.
- Classify the primary subdivisions: Kinematics, Dynamics, Statics, and Kinetics.
- Define Kinematic Links, Kinematic Joints, and Kinematic Pairs.
- Differentiate between Lower Pairs and Higher Pairs.
- Distinguish between Kinematic Chains, Mechanisms, and Machines.
- Calculate Degrees of Freedom (DOF) using Kutzbach's and Grübler's mobility criteria.
- Analyze Four-Bar Linkages and apply Grashof's Law.
- Identify inversions of single slider, double slider, and four-bar chains.
- Compare Kinematic Analysis and Kinematic Synthesis.

Definition & Scope of Theory of Machines I

Definition of Theory of Machines

Theory of Machines is that branch of Engineering Science which deals with the study of relative motion between the various parts of a machine, and the forces which act on them.

- **Bridge between Theory and Design:** Connects fundamental principles of physics and classical mechanics with practical mechanical system design.
- **Definition of a Machine:** A combination of resistant bodies so arranged that by their means the mechanical forces of nature can be compelled to do work accompanied by certain determinate motions.
- **Primary Objectives of Machine Design:**
  - *Kinematic Design:* Determining the configuration, velocities, and accelerations of all machine components.

- *Dynamic Design:* Evaluating static and dynamic forces, inertia forces, power transmission efficiency, and stress distributions.

Subdivisions of Theory of Machines I

Theory of Machines is broadly categorized into two major fields:

1. Kinematics of Machines (Kinematics)

- Deals with the **relative motion** between machine parts (displacement, velocity, and acceleration).
- Analyzes geometry of motion **without considering** the forces causing the motion.

Subdivisions of Theory of Machines II

2. Dynamics of Machines

- Deals with the **forces** acting on machine components and their resulting effects.
- Subdivided into two specialized branches:
  - **Statics:** Analyzes forces acting on machine parts when mass effects/inertia forces are neglected (parts at rest or moving at constant velocity).
  - **Kinetics:** Analyzes inertia forces resulting from the combined effects of mass and acceleration of machine parts.

Kinematic Link and Joint I

- **Kinematic Link (Element):** A resistant body (or an assembly of resistant bodies) forming part of a machine and having relative motion with another part.
- **Resistant Body Requirement:** Must not deform permanently under transmitted forces (can be rigid, flexible, or fluid).
- **Types of Links:**
  - *Rigid Link:* Negligible deformation (e.g., connecting rod, crank).

Kinematic Link and Joint II

- *Flexible Link:* Deforms in a permissible way to transmit motion (e.g., belts, ropes, chains).
- *Fluid Link:* Motion transmitted through fluid pressure (e.g., hydraulic oil in presses).

- **Kinematic Joint:** Connection between two links allowing specific relative motion.

Classification of Kinematic Pairs I

Kinematic Pair

Two kinematic links in contact having relative motion constrained in a definite direction form a kinematic pair.

Parameter & **Lower Pair** & **Higher Pair**
Contact Type & Surface or Area contact & Line or Point contact
Friction & Wear & Higher friction; distributed contact pressure & Lower friction; higher localized contact stress
Manufacturing & Easier to machine accurately & Requires complex profile manufacturing
Examples & Revolute (pin), Slider, Screw, Cylindrical pairs & Cam-follower, Gear teeth mesh, Ball bearings

Table: Comparison between Lower Pairs and Higher Pairs

Classification by Motion and Constraint I

Classification Based on Relative Motion

- **Sliding Pair (Prismatic - P):** Relative translation along a straight line (e.g., piston in cylinder).
- **Turning Pair (Revolute - R):** Pure rotation about a fixed axis (e.g., shaft in journal bearing).
- **Rolling Pair:** Pure rolling without slipping (e.g., ball/roller bearing).
- **Screw Pair (Helical - H):** Translation coupled linearly with rotation via threads (e.g., lead screw & nut).
- **Spherical Pair (S):** 3D rotational freedom (e.g., ball and socket joint).

Classification by Motion and Constraint II

Classification Based on Mechanical Constraint

- **Closed Pair (Self-closed):** Kinematically held together by geometric shape/form.
- **Unclosed Pair (Force-closed):** Held in contact by external forces like springs or gravity (e.g., cam-follower).

Kinematic Chains, Mechanisms, and Machines I

Hierarchy of Motion Systems

Kinematic Chain: Combination of kinematic links connected by kinematic pairs such that relative motion between links is completely or successfully constrained.

Mechanism: Formed when **one link** of a kinematic chain is fixed to the frame.

- *Simple Mechanism:* Consists of exactly 4 links.
- *Compound Mechanism:* Consists of more than 4 links.

Machine: A combination of mechanisms designed to transmit and modify energy to perform useful mechanical work.

Feature & **Mechanism** & **Machine**
Primary Function & Transmits and modifies motion & Transmits motion and does mechanical work
Energy Role & Focuses on kinematics & Modifies available energy into work
Examples & Clockwork, drafting machine & Lathe, IC engine, steam engine

Table: Key Differences between Mechanism and Machine

Degrees of Freedom & Mobility Criteria I

Degrees of Freedom (DOF or Mobility, F)

Number of independent coordinates or input parameters required to completely define the configuration of a mechanism.

Kutzbach's Criterion for Planar Mechanisms

For a planar mechanism with N links: $F = 3(N-1) - 2J_1 - J_2$
where:

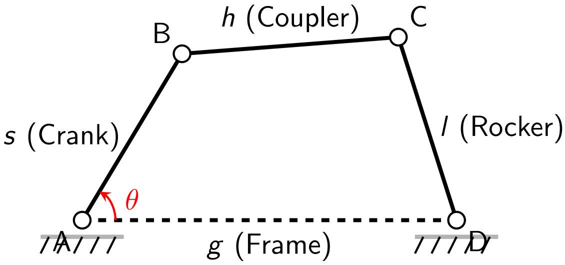
- $N =$ Total number of links (including fixed frame)
- $J_1 =$ Number of 1-DOF joints (lower pairs: revolute, sliding)
- $J_2 =$ Number of 2-DOF joints (higher pairs: cam contact, gear mesh)

Degrees of Freedom & Mobility Criteria II

Grübler's Criterion

For a single DOF mechanism ($F = 1$) containing only 1-DOF lower pairs ($J_2 = 0$):
$3(N-1) - 2J_1 = 1 \implies 3N - 2J_1 - 4 = 0 \implies J_1 = \frac{3N-4}{2}$
Implication: For J_1 to be an integer, N must be an even number ($N = 4, 6, 8, \dots$).

The Four-Bar Linkage & Grashof's Law I



Grashof's Law

For a planar four-bar linkage, if the sum of the shortest (s) and longest (l) links is less than or equal to the sum of the other two links ($p + q$), then at least one link can make a complete 360° revolution relative to the frame: $s + l \leq p + q$

Inversions of Kinematic Chains I

Concept of Inversion

The process of obtaining different mechanisms by fixing different links of a kinematic chain one at a time. The relative motion between links remains unchanged, but absolute motions relative to the fixed link change.

- **Inversions of Four-Bar Chain ($s + l \leq p + q$):**
  1. *Crank-Rocker Mechanism:* Shortest link s adjacent to fixed link.
  2. *Double-Crank Mechanism:* Shortest link s is fixed (grounded).
  3. *Double-Rocker Mechanism:* Link opposite to shortest link s is fixed.
- **Inversions of Single Slider-Crank Chain:**
  - Reciprocating Engine (Link 1 fixed)
  - Whitworth Quick Return Motion / Rotary Engine (Link 2 fixed)
  - Oscillating Cylinder Engine (Link 3 fixed)
  - Hand Pump / Bull Engine (Link 4 fixed)

Inversions of Kinematic Chains II

- **Inversions of Double Slider-Crank Chain:**
  - Elliptical Trammel, Scotch Yoke, Oldham's Coupling.

Kinematic Analysis vs. Kinematic Synthesis I

Aspect & **Kinematic Analysis** & **Kinematic Synthesis**
Objective & Analyze motion of a given mechanism & Design a mechanism for desired motion
Given Inputs & Link dimensions, configuration, input velocity/acceleration & Desired motion trajectory, path, or output function
Output & Positions, velocities, accelerations & Link lengths, joint types, mechanism layout
Uniqueness & Unique output solution & Multiple valid design solutions possible

Table: Comparison of Analysis and Synthesis

Phases of Kinematic Synthesis

1. **Type Synthesis:** Selecting the suitable mechanism type (cams, gear trains, four-bar linkages).
2. **Number Synthesis:** Determining the number of links and joints for required DOF.
3. **Dimensional Synthesis:** Calculating exact link lengths and pivot coordinates.

Kinematics of Machines: Fundamental Concepts I

Introduction to Kinematics

- **Kinematics** is the branch of machine theory that studies the relative motion between machine components (displacement, velocity, acceleration) without considering the forces causing the motion.
- Contrast with **Kinetics/Dynamics**, which explicitly analyzes forces, torques, and masses producing motion.

Course Context & Scope

- **Course Code**: DI03000141 (Theory of Machines and Mechanisms).
- **Key Focus Areas**: Kinematic links, pairs, degrees of freedom (mobility), kinematic chains, Grashof's law, and kinematic inversions.

Kinematic Links or Elements I

Definition of a Kinematic Link

A **kinematic link** (or element) is a resistant body or assembly of rigid bodies that possesses relative motion with respect to other components in a mechanism.

- **Resistant Body Requirement**: A body capable of transmitting required forces with negligible deformation under standard operational loads.

Kinematic Links or Elements II

Classification Based on Deformability

- **Rigid Link**: Experiences no detectable deformation while transmitting force (e.g., crank, connecting rod, piston, cylinder).
- **Flexible Link**: Undergoes controlled, functional deformation while transmitting motion or force in tension (e.g., belts, ropes, chains).
- **Fluid Link**: Transmits force or motion through fluid pressure under compression (e.g., hydraulic jacks, hydraulic brakes, hydraulic ram).

Kinematic Links or Elements III

Classification Based on Nodes

- **Binary Link**: Possesses 2 connection nodes.
- **Ternary Link**: Possesses 3 connection nodes.
- **Quaternary Link**: Possesses 4 connection nodes.

Classification of Kinematic Pairs: Relative Motion I

Kinematic Pair Definition

A **kinematic pair** is formed when two kinematic links are joined together such that their relative motion is completely or successfully constrained.

Table: Classification of Kinematic Pairs by Relative Motion

Pair Type & **Nature of Relative Motion** & **DoF Allowed** & **Engineering Examples**
Sliding (Prismatic) & Pure translatory relative sliding & 1 & Piston in cylinder, lathe tailstock
Turning (Revolute) & Pure rotational turning about an axis & 1 & Crankshaft in journal bearing
Rolling & Pure rolling without sliding & 1 & Ball bearings, roller bearings
Screw (Helical) & Coupled rotational and translatory motion & 1 & Lead screw and nut of lathe
Spherical & Rotational motion about a single 3D point & 3 & Ball and socket joint, mirror mount

Kinematic Pairs: Lower vs. Higher Pairs

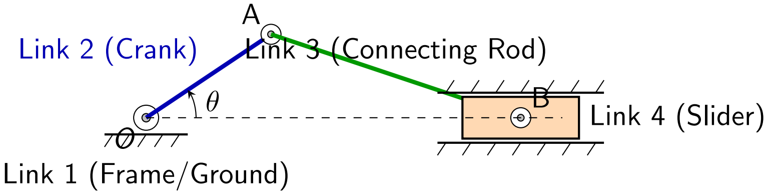
Contact-Based Pair Classification

Kinematic pairs are broadly categorized based on the contact geometry between mating link surfaces.

Table: Comparison of Lower Pairs and Higher Pairs

Parameter & Lower Pair & Higher Pair
Contact Geometry & Surface or area contact & Point or line contact
Stress Distribution & Uniform, low surface stress & High localized Hertzian contact stress
Relative Motion & Sliding, turning, or helical motion & Rolling or combined rolling-sliding
Lubrication & Easy to maintain thick fluid film & Requires high-pressure elastohydrodynamic film
Wear Behavior & Uniform wear across mating surfaces & Concentrated wear along contact lines/points
Typical Examples & Revolute joints, prismatic sliders & Cam-follower, spur gear mesh, ball bearing

Schematic of a Slider-Crank Mechanism I



Schematic of a Slider-Crank Mechanism II

Kinematic Breakdown

- **Link 1 (Frame/Ground)**: Fixed reference system.
- **Link 2 (Crank)**: Rotates continuously about fixed center O.
- **Link 3 (Connecting Rod)**: Coupler link connecting pin A to slider pin B.
- **Link 4 (Slider)**: Reciprocates linearly along the guide rails of Link 1.
- **Joint Composition**: 3 Revolute joints (at O, A, B) and 1 Prismatic joint (between slider and guide).

Degrees of Freedom & Mobility of Planar Mechanisms I

Concept of Mobility (F)

Mobility (F) represents the number of independent coordinates required to uniquely specify the configuration of a mechanism in space.

Degrees of Freedom & Mobility of Planar Mechanisms II

Kutzbach Criterion for Planar Mechanisms

For a planar mechanism operating in a 2D plane:

$$F = 3(N - 1) - 2P_1 - P_2$$

where:

- N = Total number of links in the mechanism (including ground link).
- P_1 = Number of 1-DoF kinematic joints (revolute or prismatic pairs).
- P_2 = Number of 2-DoF kinematic joints (higher pairs with rolling-sliding).

Degrees of Freedom & Mobility of Planar Mechanisms III

Gruebler's Criterion for 1-DoF Mechanisms

For a single-degree-of-freedom mechanism ($F = 1$) containing only 1-DoF joints ($P_2 = 0$):

$$3(N-1) - 2P_1 = 1 \implies 3N - 2P_1 - 4 = 0$$

This establishes that a constrained single-input mechanism requires a specific combination of link count and joint count.

Kinematic Chains vs. Mechanisms I

Kinematic Chain

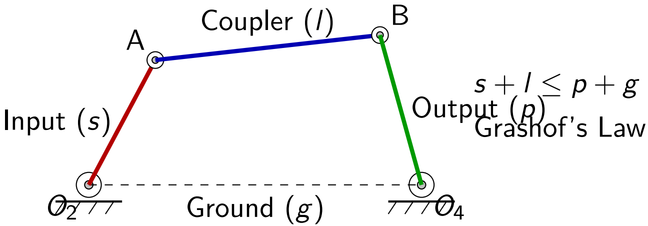
A **kinematic chain** is an assembly of links connected by kinematic pairs such that relative motion between links is predictable and repeatable.

Kinematic Chains vs. Mechanisms II

Classification Based on Mobility (F)

- **Locked Chain / Structure ($F \leq 0$):** No relative motion between links is possible.
  - If $F = 0$: Statically determinate frame or truss (e.g., 3-bar rigid triangle).
  - If $F < 0$: Statically indeterminate structure with redundant constraints.
- **Constrained Chain / Mechanism ($F = 1$):** Giving a specific input motion to one link produces a unique, predictable output motion at all other links (e.g., standard 4-bar linkage).
- **Unconstrained Chain ($F \geq 2$):** Requires two or more independent inputs to control the output configuration deterministically (e.g., 5-bar linkage with $F = 2$).

Four-Bar Linkage & Grashof's Law I



Grashof's Criterion Statement

For a planar four-bar linkage, continuous relative rotation of at least one link is possible if and only if the sum of the shortest (s) and longest (l) link lengths is less than or equal to the sum of the remaining two link lengths (p, g):

$$s + l \leq p + g$$

Grashof Linkage Behavior

- **Class I ($s + l \leq p + g$)**: At least one link can perform a full 360° rotation relative to the fixed frame.
- **Class II ($s + l > p + g$)**: Non-Grashof linkage. No link can execute a full 360° revolution; all links act as rockers.

Kinematic Inversions: Four-Bar Linkage I

Concept of Kinematic Inversion

Kinematic inversion is the process of obtaining different functional mechanisms by fixing different links of a single kinematic chain.

- **Invariant**: Relative motion between links remains identical across all inversions.
- **Variant**: Absolute motion of links relative to the ground reference frame changes significantly.

Kinematic Inversions: Four-Bar Linkage II

Inversions of the Grashof Four-Bar Chain ($s + l \leq p + g$)

- **Crank-Rocker Mechanism**: Shortest link s is adjacent to fixed link g. Link s makes full 360° rotation while opposite link rocks. (e.g., windshield wiper, sewing machine treadle).
- **Double-Crank (Drag-Link) Mechanism**: Shortest link s is fixed to ground ($g = s$). Both side links perform continuous 360° rotations. (e.g., quick return drive linkage).
- **Double-Rocker Mechanism**: Shortest link s is the coupler link (opposite to fixed link). Both grounded links oscillate/rock. (e.g., automotive ackermann steering mechanism).

Inversions of Single & Double Slider-Crank Chains I

Single Slider-Crank Chain Inversions

- **Fixed Frame (Link 1)**: Standard Reciprocating IC Engine, Air Compressor (rotary crank $\leftrightarrow$ reciprocating slider).
- **Fixed Crank (Link 2)**: Whitworth Quick Return Mechanism, Rotary Internal Combustion Engine (Gnome engine).
- **Fixed Connecting Rod (Link 3)**: Oscillating Cylinder Engine, Crank and Slotted Lever Quick Return Mechanism.
- **Fixed Slider (Link 4)**: Hand Pump (Bull Engine).

Inversions of Single & Double Slider-Crank Chains II

Double Slider-Crank Chain Inversions

- **Elliptical Trammel** (Slotted plate fixed): Points on the connecting link trace true mathematical ellipses.
- **Scotch Yoke Mechanism** (Slider or slotted link fixed): Converts linear simple harmonic motion into uniform rotary motion.
- **Oldham's Coupling** (Intermediate floating link fixed): Transmits rotation and torque between parallel non-coaxial shafts.

Summary of Machine Kinematics I

Core Conceptual Synthesis

- **Kinematic Link**: Resistant element capable of transmitting forces with negligible deformation (Rigid, Flexible, Fluid).
- **Kinematic Pair**: Joint permitting constrained motion, categorized by motion type (sliding, turning, rolling, screw, spherical) and contact geometry (Lower vs Higher).
- **Mobility Criterion**: Governed by Kutzbach's equation $F = 3(N-1) - 2P_1 - P_2$, determining independent input parameters.
- **Grashof's Condition**: $s + l \leq p + g$ dictates whether continuous full rotation is physically possible.
- **Kinematic Inversions**: Yields diverse engineering mechanisms from a single linkage chain by changing the fixed reference frame.

Introduction to Kinematics of Machines I

- **Course Context:** Theory of Machines and Mechanisms (DI03000141).
- **Definition:** Kinematics of machines is the study of the relative motion between various machine components without considering the forces causing that motion.
- **Primary Objectives:**
  - Analyze displacement, velocity, and acceleration of links.
  - Understand geometric synthesis of mechanism links.
  - Study planar and spatial kinematic arrangements.
- **Core Concepts Covered:**
  - Kinematic links and resistant bodies.
  - Kinematic pairs and lower/higher pair classifications.
  - Mobility criteria (Kutzbach and Grubler).
  - Four-bar linkages and Grashof's Law.
  - Kinematic inversions of single and double slider-crank chains.

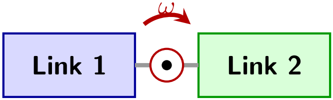
Kinematic Links and Resistant Bodies I

- **Kinematic Link (or Element):** A resistant body or assembly of rigid bodies that forms part of a machine and possesses relative motion with respect to other components.
- **Resistant Body Concept:**
  - Must transmit required forces and motion with negligible internal deformation during operation.
  - Examples: Hydraulic fluid columns under pressure, steel belts, rigid connecting rods.
- **Classification of Kinematic Links:**
  1. **Rigid Links:** Undergo no appreciable deformation under working loads (e.g., engine crankshaft, connecting rod).
  2. **Flexible Links:** Deform in a controlled, functional manner while transmitting power under tension (e.g., drive belts, ropes, chains, springs).
  3. **Fluid Links:** Transmit power through fluid hydrostatic pressure (e.g., hydraulic ram fluid, automobile braking fluid).

Kinematic Pairs and Contact Classification I

- **Kinematic Pair:** A joint formed by two constrained kinematic links that dictates their relative motion.
- **Classification by Nature of Contact:**
  - **Lower Pairs:** Relative motion occurs over a surface or area contact.
    - Distributes surface pressure, reducing wear.
    - Examples: Revolute (turning), Prismatic (sliding), Screw (helical) joints.
  - **Higher Pairs:** Contact occurs along a line or at a single point.
    - High local stress concentrations.
    - Examples: Cam and follower, ball and roller bearings, meshing gear teeth.
- **Classification by Joint Closure:**
  - **Closed (Self-Closed) Pairs:** Held together mechanically by geometry and structural enclosure.
  - **Open (Force-Closed) Pairs:** Contact is maintained via external forces (e.g., gravity or spring preload).

Revolute Joint Diagram and Lower Pairs I



Pin Joint (Revolute / 1 DOF)

- **Revolute Pair (Pin Joint):**
  - Permits pure relative rotation about a single axis.
  - Provides 1 degree of freedom ($F = 1$).
  - Surface-to-surface cylindrical contact makes it a classic **lower pair**.

Kinematic Chains and Mobility Criteria I

- **Kinematic Chain:** An assembly of kinematic pairs connected such that each link forms a part of two or more pairs, creating a closed loop.
- **Mobility (F):** Total number of independent coordinates required to define the system configuration completely.
- **Kutzbach Mobility Criterion (Planar Linkages):**

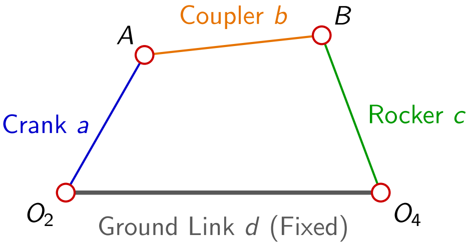
$$F = 3(n-1) - 2j - h$$

  where n = number of links, j = number of binary lower pairs, h = number of higher pairs.
- **Grubler's Criterion:** For a single degree of freedom ($F = 1$) planar mechanism with lower pairs only ($h = 0$):

$$3n - 2j - 4 = 0$$

Kinematic Chains and Mobility Criteria II

- **Physical Interpretation of F:**
  - $F > 0$: Unconstrained mechanism (F independent inputs required).
  - $F = 0$: Statically determinate structure (frame).
  - $F < 0$: Statically indeterminate structure (pre-stressed frame).



- **Grashof's Criterion for Continuous Rotation:**

$$s + l \leq p + q$$

where $s =$ shortest link, $l =$ longest link, and $p, q =$ intermediate links.

- **Mechanisms Obtained (Class I Linkages):**

Planar Four-Bar Linkage and Grashof's Law II

- **Crank-Rocker:** Shortest link adjacent to fixed link.
- **Double-Crank (Drag-Link):** Shortest link is fixed to ground.
- **Double-Rocker:** Link opposite to shortest link is fixed to ground.

Kinematic Inversions of Linkages I

- **Definition:** Kinematic inversion is the technique of creating different functional mechanisms from a single kinematic chain by choosing a different link to be fixed relative to ground.
- **Fundamental Principles:**
  - Inversion changes absolute motions relative to the frame.
  - Relative motion between individual links in the chain remains unchanged.
- **Engineering Application (Windshield Wiper System):**
  - Uses a Crank-Rocker inversion of a 4-bar linkage.
  - Converts high-speed continuous rotational input from an electric motor into smooth, wide-angle oscillatory sweeping motion.

Single Slider-Crank Chain and Inversions I

- **Chain Structure:** Consists of 4 links connected by 3 revolute (turning) pairs and 1 prismatic (sliding) pair ($n = 4, j = 4, F = 1$).
- **Four Kinematic Inversions:**
  1. **1st Inversion (Fixed Frame / Cylinder):**
     - Reciprocating IC engine, reciprocating compressor.
     - Converts sliding motion to rotary motion or vice versa.
  2. **2nd Inversion (Fixed Crank):**
     - Whitworth quick-return mechanism, Gnome rotary aircraft engine.
  3. **3rd Inversion (Fixed Connecting Rod):**
     - Crank and slotted-lever quick return mechanism, oscillating cylinder engine.
  4. **4th Inversion (Fixed Slider):**
     - Hand pump (Bull engine mechanism).

Double Slider-Crank Chain and Inversions I

- **Chain Structure:** Consists of 4 links connected by 2 revolute (turning) pairs and 2 prismatic (sliding) pairs.
- **Three Key Inversions:**
  1. **Elliptical Trammel (Fixed Slotted Frame):**
     - Used for drawing ellipses.
     - Any point on the connecting link traces an exact ellipse.
  2. **Scotch Yoke Mechanism (Fixed Slider):**
     - Converts uniform rotary motion into exact simple harmonic motion (SHM).
  3. **Oldham's Coupling (Fixed Intermediate Link):**
     - Transmits power between two parallel shafts with lateral misalignment.

Kinematic Analysis: Velocity and Coriolis Acceleration I

- **Relative Velocity Vector:**
  - Relative velocity of point B with respect to A on a rigid link:

$$\vec{v}_{BA} = \vec{\omega} \times \vec{r}_{BA}$$

  - Vector direction is always perpendicular to link AB.
- **Coriolis Acceleration Component (a^c):**
  - Present when a slider block moves with sliding velocity v along a link rotating at angular velocity ω.
  - Magnitude:

$$a^c = 2\omega v$$

  - Direction: Sliding velocity vector $\vec{v}$ rotated by 90° in the direction of link rotation ω.
- **Transmission Angle (γ):**
  - Angle between coupler link and output rocker.
  - Ideal range: $45^\circ \leq \gamma \leq 135^\circ$ to prevent mechanical binding.

Kinematic Chain	Lower Pairs	Higher Pairs	Key Inversions & Applications
Four-Bar Chain	4 Revolute	0	Crank-Rocker, Drag-Link, Wiper Linkage
Single Slider-Crank	3 Revolute, 1 Sliding	0	IC Engine, Whitworth, Slotted Lever, Hand Pump
Double Slider-Crank	2 Revolute, 2 Sliding	0	Elliptical Trammel, Scotch Yoke, Oldham's Coupling
Cam-Follower Joint	1 Turning/Sliding	1 Higher	Engine Valve Control System

Table: Summary of Kinematic Pairs, Chains, and Inversions

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit:** Motions & Mechanisms
- **Topic:** Kinematic Inversions of Four-Bar, Single Slider-Crank, and Double Slider-Crank Chains

Lecture 5: 1.3 Inversions of Chains II

Learning Objectives

- Understand the fundamental concept and physical significance of kinematic inversion.
- Analyze the inversions of a planar four-bar chain governed by Grashof's Law.
- Examine the four inversions of a single slider-crank mechanism and their engineering applications.
- Classify the inversions of a double slider-crank chain (Elliptical Trammel, Scotch Yoke, Oldham's Coupling).

Concept of Kinematic Inversion I

Definition

Kinematic inversion is the process of obtaining different mechanisms from the same kinematic chain by fixing different links, one at a time.

- **Number of Inversions:** A kinematic chain with n links can yield up to n distinct inversions by fixing each of the n links in turn.
- **Relative Motion Invariance:** The relative motion between any two links in a kinematic chain remains completely unchanged during inversion.
- **Absolute Motion Transformation:** The absolute motions of the links (relative to the stationary frame/fixed link) change significantly depending on which link is grounded.

Concept of Kinematic Inversion II

- **Engineering Illustration:** In a single slider-crank chain, fixing the cylinder yields a reciprocating engine, whereas fixing the crank yields a Whitworth quick return mechanism.

Inversions of the Four-Bar Kinematic Chain I

Four-Bar Chain Fundamentals

A basic planar four-bar chain consists of four rigid links connected by four revolute (turning) pairs.

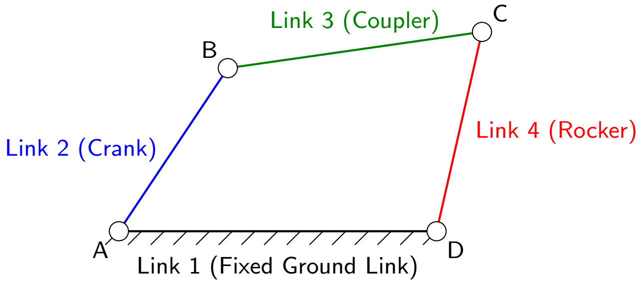
Grashof's Law

For a planar four-bar linkage, continuous relative motion between links will occur if and only if the sum of the shortest (s) and longest (l) link lengths is not greater than the sum of the remaining two link lengths (p and q):

$$s + l \leq p + q$$

- **Inversion 1: Crank-Rocker Mechanism** Fixed link is adjacent to the shortest link. The shortest link acts as the crank (completing a full 360° rotation), while the opposite link oscillates (rocker).
- **Inversion 2: Double-Crank / Drag-Link Mechanism** The shortest link is fixed. Both links adjacent to the shortest link act as cranks and undergo full 360° continuous rotations.
- **Inversion 3: Double-Rocker Mechanism** The link opposite to the shortest link is fixed. Both adjacent links act as rockers and only oscillate.

Schematic Representation of a Four-Bar Linkage I



- **Link 1 (AD):** Grounded/fixed frame serving as the reference coordinate system.
- **Link 2 (AB):** Input driving crank capable of complete 360° rotation under Grashof's condition.
- **Link 3 (BC):** Floating coupler link that transmits forces and motion from input to output.

Schematic Representation of a Four-Bar Linkage II

- **Link 4 (CD):** Output rocker/follower link oscillating between predefined angular limits.

Inversions of Single Slider-Crank Chain I

Kinematic Pair Configuration

A single slider-crank chain comprises four links connected by three turning (revolute) pairs and one sliding (prismatic) pair.

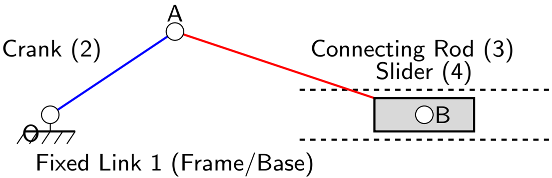
Inversion & **Fixed Link** & **Practical Application / Mechanism**
1st & Link 1 (Cylinder / Frame) & Reciprocating IC Engine, Compressor
2nd & Link 2 (Crank) & Whitworth Quick Return, Gnome Engine
3rd & Link 3 (Connecting Rod) & Crank & Slotted Lever, Oscillating Cylinder
4th & Link 4 (Slider) & Hand Pump (Bull Engine), Pendulum Pump

- **First Inversion:** Converts rotary motion into linear reciprocating motion (or vice versa).
- **Second Inversion:** Converts uniform rotary motion of one link into variable rotary motion of another.

Inversions of Single Slider-Crank Chain II

- **Third Inversion:** Yields oscillatory motion of a lever or cylinder body.
- **Fourth Inversion:** Converts oscillating lever action into reciprocating casing displacement.

Schematic of a Single Slider-Crank Mechanism I



- **Link 1 (Frame):** Stationary cylinder casing and guide ways.
- **Link 2 (Crank):** Rotates around main bearing axis at fixed pivot O.
- **Link 3 (Connecting Rod):** Pins crankpin A to gudgeon pin B.
- **Link 4 (Slider):** Executes linear translation along the horizontal axis inside the guide.

Whitworth Quick Return Mechanism

- **Configuration:** Link 2 (Crank) is grounded/fixed.
- **Application:** Industrial shaping and slotting machine tools.
- **Operating Principle:** The driving crank rotates at uniform angular velocity, causing the slotted bar to rotate at non-uniform angular velocity.
- **Advantage:** Significantly reduces non-productive idle time during the return stroke of the cutting ram.

Crank Fixed: Whitworth & Gnome Engine II

Rotary Internal Combustion Engine (Gnome Engine)

- **Configuration:** Central crankshaft (Link 2) remains stationary.
- **Application:** Early 20th-century aviation engines.
- **Mechanism Operation:** The cylinders rotate around the fixed crank while pistons reciprocate inside them.
- **Advantage:** Cylinder rotation acts as an integrated flywheel and provides rapid air cooling without additional cooling hardware.

Connecting Rod Fixed: Crank and Slotted Lever I

Crank and Slotted Lever Quick Return Mechanism

- **Configuration:** Link 3 (Connecting Rod) is fixed as the ground frame.
- **Motion Characteristics:** As the driving crank rotates continuously, a slider mounted on the crankpin slides within a slotted lever, causing the lever to oscillate.
- **Output Transverse Linkage:** The top of the slotted lever drives a horizontal ram through an articulated coupling rod.
- **Time Ratio:** The forward (cutting) stroke takes place through a larger crank rotation angle α, while the return stroke occurs over a smaller angle β ($\alpha > \beta$).

Connecting Rod Fixed: Crank and Slotted Lever II

Oscillating Cylinder Engine

- **Configuration:** Connecting rod (Link 3) is fixed.
- **Operation:** The steam cylinder oscillates back and forth about a trunnion pivot to align inlet/outlet ports while the piston rod directly turns the crankpin.

Inversions of Double Slider-Crank Chain I

Chain Architecture

Contains four links connected by two revolute (turning) pairs and two prismatic (sliding) pairs.

- **Inversion 1: Elliptical Trammel**
  - *Fixed Link:* Slotted frame containing two mutually perpendicular guides.
  - *Kinematic Result:* Any point on the link connecting the two sliders traces a perfect ellipse. The midpoint traces a circle.
- **Inversion 2: Scotch Yoke Mechanism**
  - *Fixed Link:* One of the sliding blocks.
  - *Kinematic Result:* Converts uniform rotary motion into exact simple harmonic (sinusoidal) linear reciprocating motion.
- **Inversion 3: Oldham's Coupling**
  - *Fixed Link:* Intermediate cross-head connecting link.

Inversions of Double Slider-Crank Chain II

- *Application:* Connects two parallel shafts whose axes are offset by a small lateral distance.
- *Torque Transmission:* Transmits constant angular velocity ratio (1 : 1) smoothly without velocity fluctuations.

Summary and Key Engineering Takeaways I

Kinematic Chain & **Fixed Element** & **Engineered Mechanism**
Four-Bar Chain & Shortest Link Adjacent & Crank-Rocker Mechanism
Four-Bar Chain & Shortest Link Fixed & Double-Crank / Drag-Link
Four-Bar Chain & Shortest Link Opposite & Double-Rocker Mechanism
Single Slider-Crank & Cylinder / Frame (Link 1) & Reciprocating Engine / Compressor
Single Slider-Crank & Crank (Link 2) & Whitworth Quick Return / Gnome Engine
Single Slider-Crank & Connecting Rod (Link 3) & Crank & Slotted Lever / Oscillating Engine
Single Slider-Crank & Slider (Link 4) & Hand Pump (Bull Engine)
Double Slider-Crank & Slotted Frame & Elliptical Trammel
Double Slider-Crank & Slider Block & Scotch Yoke Mechanism
Double Slider-Crank & Intermediate Link & Oldham's Coupling

Summary and Key Engineering Takeaways II

Key Takeaways

1. Kinematic inversion alters relative absolute motion with respect to ground without changing relative motion between links.
2. Mechanical design selection of the grounded link determines whether motion is rotary, reciprocating, or oscillating.
3. Grashof's criterion governs full rotation capabilities in four-bar linkages.

Lecture Overview & Learning Objectives I

Course Information

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit**: Motions & Mechanisms
- **Lecture 6**: 1.3 Inversions of Chains

Lecture Overview & Learning Objectives II

Learning Objectives

An exploration of kinematic inversions in four-bar, single-slider, and double-slider crank chains:

- Understand the fundamental concept of kinematic inversion and reference frames.
- Apply Grashof's Law to classify planar four-bar linkages ($s + l \leq p + q$).
- Analyze inversions of Four-Bar, Single-Slider Crank, and Double-Slider Crank chains.
- Evaluate practical engineering applications such as beam engines, quick return mechanisms, and Oldham's coupling.

Concept of Kinematic Inversion I

Definition

A **kinematic inversion** is obtained by fixing different links of a kinematic chain, one at a time.

Concept of Kinematic Inversion II

Key Principles of Inversion

- **Relative Motion Invariance**: The relative motion between any two links in the chain remains completely unchanged regardless of which link is fixed.
- **Absolute Motion Transformation**: The absolute motion of individual links relative to the ground changes as different links are fixed.
- **Number of Inversions**: The maximum number of possible inversions equals the total number of links (n) in the kinematic chain.
- **Symmetry Exception**: Some inversions may produce identical kinematic mechanisms if the chain possesses symmetrical geometry.
- **Mobility Conservation**: Inverting a chain does not alter its mobility or Degrees of Freedom (DoF).

Four-Bar Chain & Grashof's Law I

Grashof's Condition for Planar Linkages

Grashof's Law governs the rotatability of planar four-bar linkages. For at least one link to undergo a complete 360° continuous rotation, the sum of the shortest (s) and longest (l) link lengths must be less than or equal to the sum of the remaining two link lengths (p and q):

$$s + l \leq p + q$$

Four-Bar Chain & Grashof's Law II

Inversions under Grashof's Condition $(s + l \leq p + q)$

- **Double-Crank (Drag-Link)**: Shortest link (s) is fixed. Both adjacent links rotate completely (360°).
- **Crank-Rocker**: Link adjacent to shortest link (s) is fixed. Shortest link rotates continuously; opposite link oscillates.
- **Double-Rocker**: Link opposite to shortest link (s) is fixed. Both links adjacent to the fixed frame oscillate.

Non-Grashof Linkages $(s + l > p + q)$

All inversions yield **Double-Rocker** mechanisms; no link can complete a full 360° rotation.

Diagram of a Four-Bar Linkage I

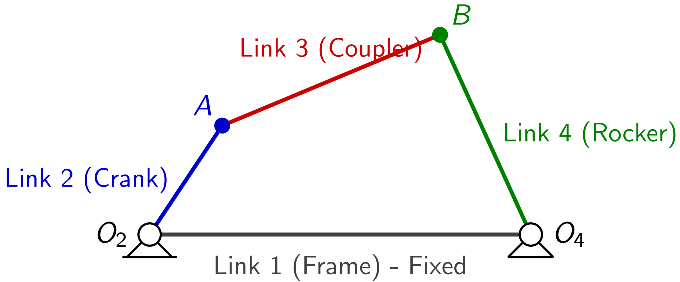


Diagram of a Four-Bar Linkage II

Kinematic Components

- **Link 1 (Frame/Ground)**: Stationary base supporting fixed pivot points O_2 and O_4.
- **Link 2 (Crank)**: Shortest link connected to input pivot O_2, performing full 360° rotation.
- **Link 3 (Coupler)**: Floating link connecting crank pin A to rocker pin B, transferring motion across the frame.
- **Link 4 (Rocker)**: Output link connected to pivot O_4, oscillating between two angular limits.

Classic Mechanisms Derived from Four-Bar Chains

Fixing distinct links yields major industrial mechanisms:

Beam Engine (Crank-Rocker) Link 1 (frame) is fixed. Converts continuous rotary motion of input crank into reciprocating motion of a vertical piston rod, commonly used in early steam pumps.

Coupling Rod of Locomotive (Double-Crank) Frame connecting two equal wheel centers is fixed. Equal length crank links ensure synchronized, parallel rotary transmission between driving wheels.

Watt's Indicator Mechanism (Double-Rocker) Link opposite to coupler is fixed. Utilizes double-rocker proportions to trace an approximate straight line, historically used in engine pressure indicators.

Single Slider Crank Chain I

Kinematic Structure

A single slider crank chain is a variant of the basic four-bar chain where one revolute (turning) pair is replaced by a prismatic (sliding) pair.

- **Pair Composition**: 3 turning pairs + 1 sliding pair.
- **Derivation**: Conceptually derived by extending one rocker link to infinite length.

Single Slider Crank Chain II

Link Definitions in Reciprocating Engine

1. **Link 1 (Frame/Cylinder Guide)**: Stationary reference guide structure.
2. **Link 2 (Crank)**: Rotating link connected to crankshaft center.
3. **Link 3 (Connecting Rod)**: Intermediate link transmitting motion between crank pin and crosshead pin.
4. **Link 4 (Crosshead/Piston)**: Sliding block undergoing linear translation.

Inversions of Single Slider Crank Chain I

Summary of Four Inversions

Different functional mechanisms emerge depending on which of the four links is immobilized.

Inversion	Fixed Link	Typical Applications
1st	Link 1 (Cylinder Guide)	Reciprocating IC Engines, Air Compressors
2nd	Link 2 (Crank)	Whitworth Quick Return, Gnome Rotary Engine
3rd	Link 3 (Connecting Rod)	Crank & Slotted Lever, Oscillating Engine
4th	Link 4 (Slider/Piston)	Hand Pump (Plunger Pump), Bull Engine

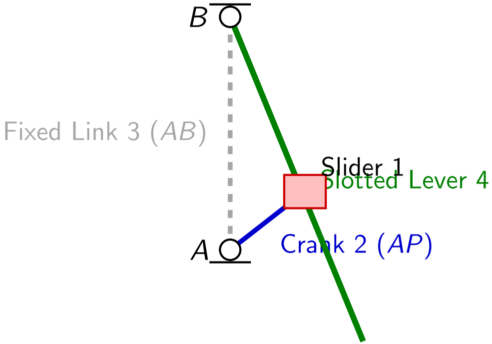
Table: Inversions of Single Slider Crank Chain

Inversions of Single Slider Crank Chain II

Key Insights

- Fixing the **Crank** or **Connecting Rod** converts pure continuous rotation into non-uniform angular oscillation, forming quick return drives.
- Fixing the **Slider** forces the cylinder to oscillate or reciprocate relative to the piston.

Crank & Slotted Lever Quick Return Mechanism I



Mechanism Operation

- Derived from the **Third Inversion** of single slider crank chain (Connecting Rod fixed).
- As Crank 2 rotates uniformly about Pivot A, Slider 1 reciprocates within the slot of Lever 4.
- Slotted Lever 4 oscillates about fixed Pivot B.
- **Quick Return Principle**: Cutting forward stroke takes place over angle $\alpha > 180^\circ$, whereas return idle stroke occurs over $\beta < 180^\circ$, maximizing machining efficiency.

Double Slider Crank Chain & Inversions I

Chain Configuration

A double slider crank chain contains 4 links arranged with **2 sliding pairs** and **2 turning pairs**. It effectively transforms linear reciprocating motions along orthogonal axes into rotational motion.

Double Slider Crank Chain & Inversions II

Inversions and Industrial Applications

1st Inversion (Slotted Frame Fixed) **Elliptical Trammel**: As sliders move in perpendicular channels, every intermediate point on the connecting link traces an exact ellipse.

2nd Inversion (One Slider Fixed) **Scotch Yoke Mechanism**: Converts uniform rotary motion into exact Simple Harmonic Motion (SHM) linear output.

3rd Inversion (Connecting Link Fixed) **Oldham's Coupling**: Used to connect two parallel shafts with a small lateral misalignment, providing constant velocity transmission.

Comparison of Kinematic Chains and Inversions I

Kinematic Chain	**Joint Composition**	**Primary Inversions**	**Motion Characteristics**
Four-Bar Chain	4 Revolute Pairs	Beam Engine, Locomotive Rod, Watt's Indicator	Rotary to Oscillating, Synchronized Rotary
Single Slider Crank	3 Revolute, 1 Prismatic Pair	IC Engine, Whitworth, Slotted Lever, Hand Pump	Rotary to Reciprocating, Quick Return stroke
Double Slider Crank	2 Revolute, 2 Prismatic Pairs	Elliptical Trammel, Scotch Yoke, Oldham's Coupling	Harmonic linear motion, Ellipse tracing, Parallel shaft coupling

Table: Comprehensive Comparison of Kinematic Chains

Engineering Applications & Summary I

Industrial Application Domains

- **Machine Tools**: Shaping, slotting, and planing machines leverage quick-return inversions to reduce non-cutting idle time.
- **Automotive & Engines**: Single-slider 1st inversion forms the basis of internal combustion power trains.
- **Power Transmission**: Oldham's coupling handles lateral shaft offset without velocity fluctuation or universal joint backlash.

Engineering Applications & Summary II

Core Takeaways

- Kinematic inversion alters the ground frame of reference without affecting relative joint motion or chain mobility.
- Grashof's criterion $(s + l \leq p + q)$ determines whether full rotational movement is physically achievable in a 4-bar loop.
- Machine design synthesis relies on selecting the appropriate inversion to achieve target trajectory and velocity profiles.

Lecture Overview: Concept of Velocity & Acceleration I

Course Details

- **Course:** DI03000141 – Theory of Machines and Mechanisms
- **Unit:** Motions & Mechanisms
- **Topic:** 1.4 Concept of Velocity and Acceleration

Lecture Overview: Concept of Velocity & Acceleration II

Key Learning Objectives

- Understand the principles of kinematic analysis without considering external forces.
- Derive linear and angular velocity relationships for rigid links.
- Decompose acceleration into radial (centripetal) and tangential components.
- Analyze the Coriolis component of acceleration, its magnitude ($2v\omega$), and directional determination.
- Evaluate kinematic equations of displacement, velocity, and acceleration in slider-crank mechanisms.

Kinematic Analysis Overview I

Definition and Scope

Kinematic analysis is the study of relative motion between machine components (links) without regard to the forces or masses causing the motion. Its primary objective is to determine displacements, velocities, and accelerations.

Kinematic Analysis Overview II

Core Kinematic Parameters

- **Displacement (x, θ):** Position change of a link or point relative to a reference frame (linear in meters or angular in radians).
- **Velocity (v, ω):** Rate of change of displacement ($v = \frac{dx}{dt}$, $\omega = \frac{d\theta}{dt}$). Defines speed and direction of link movement.
- **Acceleration (a, α):** Rate of change of velocity ($a = \frac{dv}{dt}$, $\alpha = \frac{d\omega}{dt}$). Critical for computing dynamic inertia forces ($F_{\text{inertia}} = -m \cdot a$).

Kinematic Analysis Overview III

Importance in Engineering Design

In high-speed machinery, acceleration analysis is paramount because inertia forces scale with the square of speed (ω^2). Neglecting acceleration leads to severe mechanical failure, excessive vibration, and rapid fatigue.

Linear and Angular Velocity Relations I

Motion of a Rigid Link

Consider a rigid link AB of length r rotating about a fixed pivot A with angular velocity ω (in rad/s).

Linear and Angular Velocity Relations II

Key Relations and Equations

- **Relative Linear Velocity (v_{BA}):**

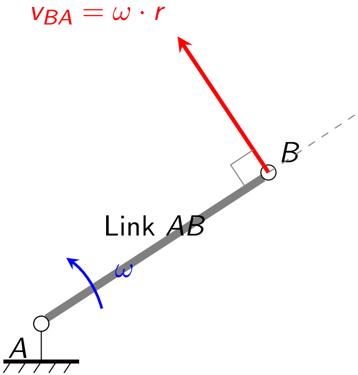
$$v_{BA} = \omega \times r$$

- **Direction:** Vector v_{BA} is always perpendicular to link AB in the sense of rotation ω.
- **Angular Velocity (ω):**

$$\omega = \frac{v_{BA}}{r} \quad \text{(rad/s)}$$

- **Rigid Body Constraint:** Every point on link AB shares the same angular velocity ω, but linear velocity varies directly with distance from pivot A.

Relative Velocity Vector in a Link I



Relative Velocity Vector in a Link II

Vector Construction Principles

- Link AB has fixed length r. Point A is the instantaneous center of rotation.
- The velocity of B relative to A (v_{BA}) is normal to line AB.
- Magnitude is proportional to radius r: $v_{BA} = \omega \cdot r$.

Concept of Acceleration in Mechanisms I

Dual Components of Acceleration

When a rigid link rotates with angular velocity ω and angular acceleration α, the acceleration of point B relative to point A consists of two orthogonal components.

1. Radial (Centripetal) Component (a_{BA}^r)

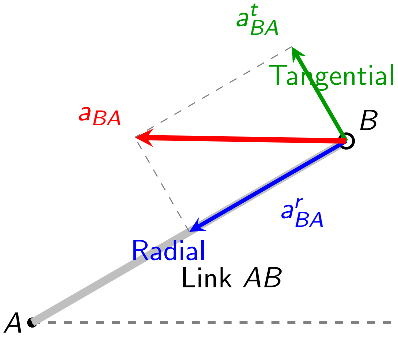
- **Cause:** Change in direction of linear velocity vector.
- **Magnitude:** $a_{BA}^r = \omega^2 \cdot r = \frac{v_{BA}^2}{r}$.
- **Direction:** Parallel to link AB, directed from B towards rotation center A.

Concept of Acceleration in Mechanisms II

2. Tangential Component (a_{BA}^t)

- **Cause:** Change in magnitude of linear velocity vector.
- **Magnitude:** $a_{BA}^t = \alpha \cdot r$, where $\alpha = \frac{d\omega}{dt}$ (rad/s^2).
- **Direction:** Perpendicular to link AB, pointing in the direction of angular acceleration α.

Acceleration Components Vector Diagram I



Acceleration Components Vector Diagram II

Resultant Acceleration Formula

The total acceleration a_{BA} is the vector sum of radial and tangential components:

$$a_{BA} = \sqrt{(a^r_{BA})^2 + (a^t_{BA})^2}$$

$$\phi = \tan^{-1}\left(\frac{a^t_{BA}}{a^r_{BA}}\right)$$

Physical Phenomenon

Coriolis acceleration occurs when a slider/point moves along a path or guide link that is simultaneously undergoing angular rotation.

Applications

Common in Whitworth quick-return mechanisms, crank and slotted-lever mechanisms, radial drilling machines, and robot manipulator arms.

Coriolis Component of Acceleration II

Mathematical Expression

$$a^c = 2 \cdot v \cdot \omega$$

where:

- v: Linear sliding velocity of the slider relative to the rotating guide link (m/s).
- ω: Angular velocity of the rotating guide link (rad/s).

Physical Origin

The factor 2 arises from two simultaneous effects:

1. Change in tangential velocity due to sliding motion changing radius ($v \cdot \omega$).
2. Rotation of the direction of sliding velocity vector in space ($\omega \cdot v$).

Determining Direction of Coriolis Acceleration I

Rule of Thumb

To find the direction of the Coriolis acceleration vector $\vec{a}^c$, rotate the sliding velocity vector $\vec{v}$ by 90° in the direction of the angular velocity $\vec{\omega}$ of the rotating link.

Vector Cross Product Notation

$$\vec{a}^c = 2(\vec{\omega} \times \vec{v})$$

Direction Determination Matrix

Case	**Sliding Motion**	**Link Rotation (ω)**	**Coriolis Direction ($\vec{a}^c$)**
1	Outwards (Away from pivot)	Counter-Clockwise (CCW)	Rotated 90° CCW (Left)
2	Outwards (Away from pivot)	Clockwise (CW)	Rotated 90° CW (Right)
3	Inwards (Towards pivot)	Counter-Clockwise (CCW)	Rotated 90° CCW (Right)
4	Inwards (Towards pivot)	Clockwise (CW)	Rotated 90° CW (Left)

Kinematic Analysis of Slider-Crank Mechanism I

Mechanism Parameters

- Crank radius $OB = r$, rotating at uniform angular speed ω.
- Connecting rod length $AB = l$.
- Obliquity ratio $n = \frac{l}{r}$ (typically between 3 and 5).
- Crank angle θ measured from inner dead center (IDC).

Kinematic Analysis of Slider-Crank Mechanism II

Kinematic Equations of Piston/Slider

- **Displacement (x):**

$$x = r\left[(1-\cos\theta) + \frac{\sin^2\theta}{2n}\right]$$

- **Velocity (v):**

$$v = r\omega\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)$$

- **Acceleration (a):**

$$a = r\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

Boundary Conditions

At Inner Dead Center ($\theta = 0^\circ$): Maximum acceleration $a_{\text{max}} = r\omega^2 \left(1 + \frac{1}{n}\right)$.

Comparison of Acceleration Components I

Summary Table of Acceleration Types

Component	Physical Cause	Magnitude	Direction
Radial (a^r)	Change in velocity direction	$\omega^2 r = \frac{v^2}{r}$	Parallel to link, towards pivot
Tangential (a^t)	Change in velocity speed	$\alpha \cdot r$	Perpendicular to link
Coriolis (a^c)	Sliding along rotating link	$2v\omega$	$\vec{v}$ rotated 90° in direction of ω

Key Takeaway

Total acceleration of any point in a general planar mechanism is computed by combining all active radial, tangential, and Coriolis components vectorially.

Summary & Practical Engineering Applications I

Summary of Core Concepts

- Kinematic velocity analysis relies on perpendicularity of relative velocity vectors for rigid links.
- Acceleration consists of centripetal ($\omega^2 r$) and tangential (αr) components.
- Coriolis acceleration ($2v\omega$) must be accounted for in mechanisms with sliding on rotating guides.

Summary & Practical Engineering Applications II

Practical Applications in Engineering

- **Dynamic Load Calculation:** High acceleration creates large inertia forces ($F = -ma$) affecting pin sizes and structural integrity.
- **Engine Balancing:** Slider-crank acceleration equations govern primary and secondary balance mass calculations in internal combustion engines.
- **Computer Simulation:** Algorithms in CAD/CAE tools (e.g., ADAMS, SolidWorks Motion) utilize these kinematic fundamentals to solve multibody dynamic systems.

1.4 Concept of Velocity and Acceleration I

Course Details

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit 1:** Kinematics of Machines
- **Topic 1.4:** Concept of Velocity and Acceleration

Lecture Objectives

To establish fundamental kinematic concepts of linear and angular displacement, velocity, and acceleration for rigid bodies and points in planar mechanisms.

Linear Displacement and Velocity I

Linear velocity is the time rate of change of linear displacement of a moving point along its trajectory.

- Consider a point P moving along a curved path in a planar coordinate system.
- Let $\mathbf{r}(t)$ represent the position vector of point P at time t.
- Instantaneous linear velocity vector: $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}$
- **Direction:** Always tangent to the path of motion at the instantaneous position of point P.
- **Magnitude:** Defined as $v = \frac{ds}{dt}$, where s is the arc length along the trajectory.

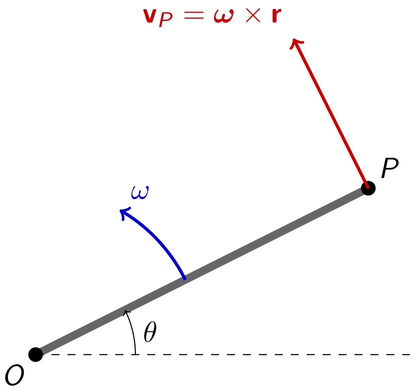
Angular Displacement and Velocity I

Angular velocity is the time rate of change of angular position of a rotating rigid link or body.

- Consider a rigid link rotating about a fixed axis or undergoing general plane motion.
- Angular displacement θ is measured in radians relative to a fixed reference axis.
- Instantaneous angular velocity: $\omega = \frac{d\theta}{dt} \quad \text{(rad/s)}$
- For a point P on a link rotating with angular velocity $\boldsymbol{\omega}$ about center O: $\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_{P/O}$
- The magnitude is given by $v_P = \omega \cdot r_{OP}$.
- **Direction:** Perpendicular to line OP in the direction of rotation.

Velocity Diagram of a Simple Link I

Graphical representation of a simple link OP rotating about a fixed center O:



Velocity Diagram of a Simple Link II

- Angular velocity ω drives the rotation, causing point P to move.
- The linear velocity vector $\mathbf{v}_P$ is strictly perpendicular to the link line OP.
- Magnitude: $v_P = \omega \cdot r_{OP}$.
- Forms the fundamental basis for constructing velocity polygons in complex mechanisms.

Linear Acceleration I

Linear acceleration is the time rate of change of linear velocity of a point.

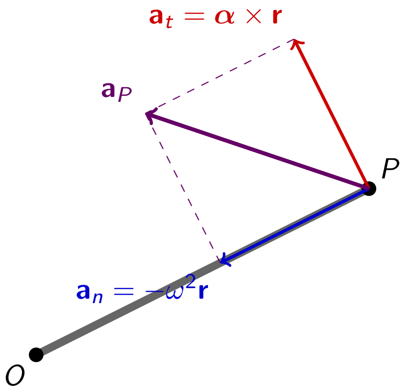
- Total acceleration vector: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$
- For motion along a curved trajectory, acceleration is decomposed into two orthogonal components:
  1. **Tangential Acceleration ($\mathbf{a}_t$):** Represents change in speed magnitude. $a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ Directed parallel (or anti-parallel) to the velocity vector.
  2. **Normal Acceleration ($\mathbf{a}_n$):** Represents change in velocity direction. $a_n = \frac{v^2}{\rho}$ where ρ is the radius of curvature. Directed inward toward the center of curvature.
- Total magnitude: $a = \sqrt{a_t^2 + a_n^2}$.

Angular Acceleration I

Angular acceleration is the time rate of change of angular velocity of a link.

- Instantaneous angular acceleration: $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad (\text{rad/s}^2)$
- For a point P at distance r from rotation center O on a link with ω and α:
  - **Tangential Acceleration Component:**
    $a_t = \alpha \cdot r \quad$ (perpendicular to link OP)
  - **Normal (Centripetal) Acceleration Component:**
    $a_n = \omega^2 \cdot r = \frac{v_P^2}{r} \quad$ (along link OP from P towards O)
- Total acceleration of point P: $\mathbf{a}_P = \mathbf{a}_t + \mathbf{a}_n$.

Vector representation of normal and tangential components of acceleration for point P:



Acceleration Components of a Rotating Link II

- Normal component $\mathbf{a}_n$ points directly from P towards rotation center O.
- Tangential component $\mathbf{a}_t$ is perpendicular to OP in direction of α.
- $\mathbf{a}_n$ always exists whenever $\omega \neq 0$, whereas $\mathbf{a}_t = 0$ if $\omega = \text{constant}$ $(\alpha = 0)$.

Relative Velocity in Mechanisms I

Relative velocity is the difference between the velocity vectors of two points on a mechanism.

- For points A and B on a rigid link: $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$ where $\mathbf{v}_{BA}$ is the relative velocity of B with respect to A.
- **Rigid Body Constraint:** Distance AB is fixed; point B can only rotate relative to A.
- **Direction:** Relative velocity $\mathbf{v}_{BA}$ must be perpendicular to line AB.
- **Magnitude:** $v_{BA} = \omega_{AB} \cdot AB$, where ω_{AB} is the angular velocity of link AB.

Relative Acceleration in Mechanisms I

Relative acceleration accounts for both relative angular velocity and angular acceleration between two points.

- Relative acceleration vector equation:
  $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} = \mathbf{a}_A + \mathbf{a}_{BA}^n + \mathbf{a}_{BA}^t$
- **Normal Relative Acceleration ($\mathbf{a}_{BA}^n$):**
  $a_{BA}^n = \frac{v_{BA}^2}{AB} = \omega_{AB}^2 \cdot AB$ (directed from B to A)
- **Tangential Relative Acceleration ($\mathbf{a}_{BA}^t$):**
  $a_{BA}^t = \alpha_{AB} \cdot AB$ (perpendicular to line AB)
- Serves as the foundation of the relative acceleration method for mechanism linkage analysis.

Coriolis Component of Acceleration I

The Coriolis component arises when a point slides along a path that is itself rotating.

- Common in mechanisms such as the Quick Return Mechanism (crank-slotted lever).
- **Magnitude of Coriolis Acceleration:** $a_c = 2 \cdot \omega \cdot v$ where ω is angular velocity of the guide link, and v is relative sliding velocity.
- **Direction Rule:** Rotate the relative sliding velocity vector $\mathbf{v}$ by 90° in the direction of the angular velocity ω.
- Critical requirement: Omitting the Coriolis component leads to severe errors in force and stress analysis of sliding pairs on rotating links.

Summary of Kinematic Concepts I

Kinematic Quantity & **Symbol** & **Formula** & **Direction / Orientation**
Linear Velocity & $\mathbf{v}$ & $\frac{d\mathbf{r}}{dt}$ & Tangent to path trajectory
Angular Velocity & ω & $\frac{d\theta}{dt}$ & Perpendicular to rotation plane
Tangential Acceleration & $\mathbf{a}_t$ & $\alpha \cdot r$ & Perpendicular to link line
Normal Acceleration & $\mathbf{a}_n$ & $\omega^2 \cdot r$ & Along link towards center
Relative Velocity & $\mathbf{v}_{BA}$ & $\omega_{AB} \cdot AB$ & Perpendicular to link AB
Coriolis Acceleration & $\mathbf{a}_c$ & $2\omega v$ & $\mathbf{v}$ rotated 90° in direction of ω

- **Course Code:** DI03000141 (Theory of Machines and Mechanisms)
- **Unit 2:** Mechanisms with Lower and Higher Pairs
- **Topic:** 2.1 Concept, Application, and Classification of Cams
- **Department:** Department of Mechanical Engineering

Course Intro: Theory of Machines and Mechanisms II

Learning Objectives

1. Understand the mechanical definition and operating principles of cam-follower systems as higher pairs.
2. Identify key industrial applications demanding precise kinematic synchronization.
3. Evaluate advantages, limitations, and tribological constraints of cam mechanisms.
4. Classify cams and followers according to geometric profiles, contact interfaces, and line of action alignment.
5. Master fundamental cam nomenclature including base circle, pitch curve, and pressure angle.

The Concept of Cams and Followers I

Definition & Kinematic Function

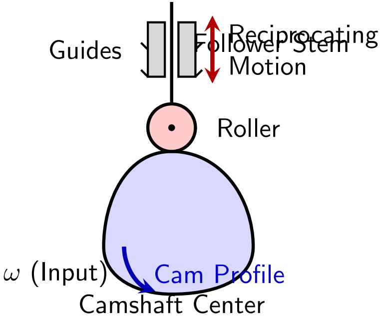
A **cam-follower mechanism** is a direct-contact, higher-pair mechanical system designed to transform a simple input motion (typically uniform rotation of the cam) into a pre-programmed, non-uniform output motion (reciprocation or oscillation of the follower).

- **Higher Pair Contact:** The interface between the cam profile and follower element features point or line contact, resulting in localized Hertzian stresses.
- **Kinematic Versatility:** Enables arbitrary displacement-time relationships ($s(t)$, $v(t)$, $a(t)$) that are difficult or impossible to synthesize using standard four-bar linkages.
- **Contact Maintenance Modes:**

The Concept of Cams and Followers II

- *Force Closure:* Contact maintained via external force elements such as return springs or gravity.
- *Form Closure:* Contact enforced geometrically via grooved profiles, conjugate cams, or desmodromic paths.

Basic Components of a Radial Cam System I



- **Cam Driver:** Rotating profile transmitting normal contact forces to the follower.
- **Follower Assembly:** Translating or oscillating element guided along a restricted kinematic axis.
- **Frame / Guides:** Fixed structural links providing line-of-action constraints for the follower stem.

Industrial Applications of Cams I

- **Internal Combustion Engine Valvetrains:**
  - Overhead camshafts (OHC) actuate intake and exhaust valves at precise crankshaft rotational angles.
  - Governs valve lift height, opening duration, and overlap to optimize volumetric efficiency and emissions.
- **Automated Assembly & Packaging Equipment:**
  - Mechanical indexing tables, automated pick-and-place grippers, and carton feeders rely on cams for zero-latency, high-repeatability motion.
- **Textile Looms & Printing Machinery:**
  - Dobby and jacquard mechanisms utilize high-speed cams to drive complex yarn shunts under heavy continuous duty.
- **Mechanical Computing & Historical Automata:**
  - Programmed drum cams functioned as non-volatile hardware storage for analog function generation before digital microcontrollers.

Advantages and Design Constraints I

Key Advantages

- **Exact Motion Control:** Capability to generate custom dwell-rise-dwell-fall profiles with precise timing.
- **Mechanical Compactness:** Requires fewer links than equivalent multi-bar linkage mechanisms.
- **High Reliability:** Fixed mechanical paths eliminate software/sensor timing jitter.

Advantages and Design Constraints II

Design Limitations & Challenges

- **Hertzian Surface Stresses:** High point/line pressure causes surface fatigue, pitting, and spalling.
- **Manufacturing Cost:** Requires high-precision CNC toolpaths and surface grinding to prevent acceleration discontinuities.
- **Dynamic Limits:** High operating speeds induce follower jump (decoupling), spring surge, and vibration.

Classification of Cams by Shape I

- **Radial (Plate) Cam:**
  - The follower moves in a plane perpendicular to the camshaft axis of rotation.
  - Simplest geometry, widely used in IC engines and light automated machinery.
- **Cylindrical (Barrel) Cam:**
  - Profile consists of a helical/grooved track cut along the perimeter of a rotating cylinder.
  - Follower moves parallel to the cylinder axis, providing positive form-closure driving action.
- **Wedge Cam:**
  - A translating wedge block with an inclined surface drives a reciprocating vertical follower.
  - Used for short-stroke applications, clamping tools, and force multiplication.
- **Globoidal & Conjugate Cams:**

Classification of Cams by Shape II

- Globoidal cams utilize a convex or concave 3D profile to drive indexing turrets across non-parallel axes.
- Conjugate cams feature dual co-rotating plates operating two follower arms to eliminate return springs entirely.

Classification of Followers by Contact Type I

Table: Comparison of Follower Contact Geometries

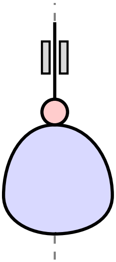
Follower Type	Advantages	Disadvantages	Primary Application
Knife-Edge	Simple geometry, precise point tracking	Extreme surface stress, rapid wear	Instrument mechanisms, low speed
Roller	Rolling friction, minimal wear, high speeds	Higher component mass, side thrust	IC engines, industrial automation
Flat-Faced	Zero side thrust in guides, compact	High sliding friction, edge loading risk	Automobile valve gear
Spherical	Prevents edge loading from deflection	Higher surface stress than flat face	Heavy-duty engine pushrods

- **Knife-Edge:** High theoretical stress limits application to delicate instruments.

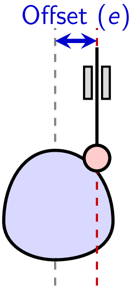
Classification of Followers by Contact Type II

- **Roller:** Standard industrial choice due to low rolling resistance and long fatigue life.
- **Flat-Faced:** Preferred when steep pressure angles might cause roller followers to bind in guides.

Follower Classification by Axis Alignment I



(a) Radial Follower
Line of action passes through cam center



(b) Offset Follower
Line of action offset by e to reduce thrust/pressure angle

- **Radial Follower:** Longitudinal axis of the follower passes through the rotational axis of the camshaft.

Follower Classification by Axis Alignment II

- **Offset Follower:** Longitudinal axis is displaced by an offset distance e. Offset reduces side thrust during the critical rise stroke, minimizing friction and binding in the guides.

Fundamental Cam Nomenclature I

- **Base Circle:** The smallest circle tangent to the physical cam profile, drawn from the center of rotation. Governs overall mechanical dimensions.
- **Trace Point:** Theoretical reference point on the follower (e.g., center of roller pin or apex of knife edge) used to compute kinematics.
- **Pitch Curve:** Path traced by the trace point if the cam profile were held stationary while the follower revolved around it.
- **Prime Circle:** Smallest circle tangent to the pitch curve centered at the cam rotation axis. For roller followers:

$$R_{\text{prime}} = R_{\text{base}} + R_{\text{roller}}$$

- **Pressure Angle (ϕ):** Angle between the normal to the pitch curve and the directional vector of follower translation.

Fundamental Cam Nomenclature II

- *Critical Design Limit:* To avoid mechanical jamming in guides, $\phi_{\mathrm{max}} \leq 30^\circ$ for translating followers.

- **Pitch Point & Pitch Circle:** Point of maximum pressure angle ϕ_{max} and circle passing through it.

Lecture Summary and Outlook I

Summary of Lecture 9

- Cam-follower systems act as higher pairs transforming uniform input rotation into customized non-uniform output motion.
- Cams are classified by physical geometry (radial, barrel, wedge, conjugate) and follower contact design (knife-edge, roller, flat-faced, spherical).
- Axis alignment (radial vs. offset) dictates side thrust forces and guide wear.
- Core geometric parameters—including Base Circle (R_b), Pitch Curve, and Pressure Angle ($\phi \leq 30^\circ$)—determine physical envelope and mechanical feasibility.

Lecture Summary and Outlook II

Outlook: Lecture 10

- Kinematic synthesis of follower displacement curves:
  - Simple Harmonic Motion (SHM)
  - Uniform Acceleration and Retardation (UARM)
  - Cycloidal Motion
- Graphical construction of cam profiles for specified follower motions.

Lecture Overview: Concept, Application, and Classification of Cams I

Course Details & Topic Context

- **Course:** GTU Course DI03000141 – Theory of Machines and Mechanisms
- **Unit II:** Cams and Followers
- **Lecture 10:** 2.1 Concept, Application, and Classification of Cams

Lecture Overview: Concept, Application, and Classification of Cams II

Primary Learning Objectives

- Master the fundamental concept of cams as higher-pair mechanical elements for non-uniform motion generation.
- Identify the core members of a cam-follower mechanism (Cam, Follower, Frame/Guide).
- Explore major industrial applications ranging from internal combustion engines to automated assembly systems.
- Classify cams according to geometry, spatial arrangement, and motion constraints.
- Categorize followers based on contact surface shape, motion type, and offset alignment.

The Basic Concept of Cams I

Definition of a Cam Mechanism

A **cam** is a rotating, oscillating, or translating machine element that imparts a predetermined, specified, and non-uniform motion to another element called the **follower** through direct contact.

The Basic Concept of Cams II

Kinematic Characteristics

- **Driving Element:** The cam acts as the driver, usually turning at a uniform angular velocity ω.
- **Driven Element:** The follower is the driven link, executing reciprocating translation or angular oscillation.
- **Kinematic Pair Type:** The interface between cam and follower forms a **higher pair** (point contact or line contact), allowing complex displacement functions $y(t)$ or $y(\theta)$.
- **Design Versatility:** Enables complex kinematic functions (e.g., Simple Harmonic Motion, Constant Acceleration, Cycloidal Motion) that are difficult or cumbersome to achieve using lower-pair linkage mechanisms alone.

Anatomy of a Cam-Follower System I

Essential Components

Every functional cam-follower mechanism relies on three basic members:

1. **Cam (Driver):** Member with a contoured profile or surface groove that dictates output kinematics.
2. **Follower (Driven):** Member that moves along specified path (translating stem or oscillating arm).
3. **Frame / Guide (Fixed Link):** Stationary link providing journal bearings or linear slide guides to constrain movement.

Anatomy of a Cam-Follower System II

Joint Closure and Force Maintenance

Because higher pairs can separate under dynamic loading, continuous contact must be maintained via:

- **Force Closure:** Using external force elements like pre-loaded compression springs, gravity, or hydraulic pressure.
- **Form (Geometric) Closure:** Utilizing grooved profiles (barrel/desmodromic cams) or dual conjugate cams where geometric constraints physically prevent disengagement.

Industrial and Engineering Applications

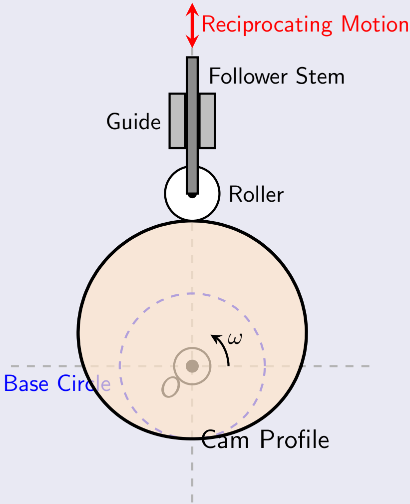
Widespread Machine Applications

Cams are irreplaceable where exact timing, precise stroke profile, and multi-axis synchronization are paramount:

- **Internal Combustion (IC) Engines:** Overhead camshafts (OHC) actuate intake and exhaust valves, controlling valve lift, timing, and dwell periods.
- **Automatic Assembly Machinery:** High-speed mechanical automation uses central camshafts to sequence feeding, clamping, indexing, and ejection.
- **Textile and Weaving Looms:** Complex double-disc or box cams govern needle and shuttle movement to produce complex fabric weaves.
- **Printing and Packaging Presses:** Precise paper feeding, gripping mechanisms, and rotary cutter actuation.
- **Mechanical Clockwork & Swaging Machines:** Precise cyclic indexing, step-by-step motion generation, and automated mechanical timers.

Geometry of a Radial Cam with Roller Follower I

Schematic Representation



Geometry of a Radial Cam with Roller Follower III

Key Anatomical Terms

- **Base Circle:** The smallest circle tangent to the cam profile drawn from cam center O.
- **Prime Circle:** Smallest circle tangent to the pitch curve (path traced by roller center).
- **Roller Follower:** Reduces surface sliding friction to pure rolling, extending tool life.

Classification of Cams Based on Shape and Geometry

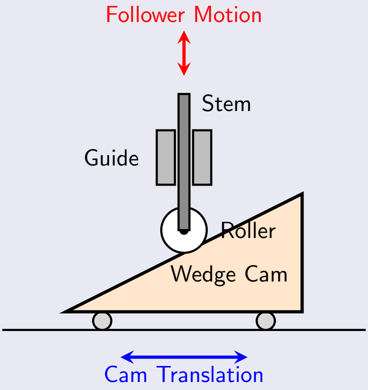
Morphological Taxonomy

Cams are categorized primarily by physical shape and relative orientation of input/output motions:

- **Radial (Disc) Cam:** Cam contour is on the outer boundary of a flat disc; follower moves perpendicular to cam's axis of rotation.
- **Cylindrical (Barrel) Cam:** Surface features a circumferential helical groove; follower stem moves parallel to the cylinder's rotational axis.
- **Wedge (Translational) Cam:** A flat plate with a profiled edge or groove that translates linearly to actuate a reciprocating follower.
- **Conjugate (Double-Disc) Cams:** Consist of two co-axial cams operating against two opposed rollers on a single follower arm to enforce positive constraint without springs.
- **Spherical / Globoidal Cam:** Curved, three-dimensional profile used to actuate oscillating levers across non-parallel rotational axes.

Kinematics of a Wedge (Translational) Cam Mechanism I

Schematic Diagram



Kinematics of a Wedge (Translational) Cam Mechanism II

Kinematic Principles

- Input translation velocity v_c converts to output follower vertical velocity v_f.
- Relation: $v_f = v_c \cdot \tan\theta$, where θ is local slope angle of wedge surface.
- Requires rigid linear guide rails to prevent skewing and jamming during stroke execution.

Classification of Followers by Contact Surface Geometry

Comparative Analysis of Follower Types

Follower Type	Contact Type	Friction & Wear	Primary Application
Knife-Edge	Point contact	Extremely high wear & contact stress	Laboratory testing, light instruments
Roller	Rolling (line)	Low friction, minimal wear	Automotive engines, heavy machinery
Flat-Faced	Line contact	Moderate wear, no side jamming	IC engine valve gear (tappets)
Spherical-Faced	Surface cap	Prevents edge loading/tilting	High-load engine valve trains

Design Considerations

- **Knife-Edge:** Traces exact theoretical motion profile, but unsuited for industrial power transmission due to sharp wear.
- **Flat-Faced:** Eliminates high contact stresses of knife-edges and prevents lateral bending forces, but requires larger base circle radius to prevent undercutting.

Classification of Followers by Motion Path and Line of Action I

Classification by Motion Type

- **Reciprocating (Translating) Follower:** Moves linearly back and forth along straight guide tracks.
- **Oscillating (Pivoted) Follower:** Pivots about a fixed fulcrum pin, converting cam profile curves into angular displacement $\theta(t)$.

Classification by Line of Action

- **Radial (In-Line) Follower:** Line of motion passes directly through cam's center of rotation O.
- **Offset Follower:** Line of motion is offset by perpendicular distance e (eccentricity) from cam center O.

Classification of Followers by Motion Path and Line of Action II

Effect of Offset e

Offsetting the follower reduces the **pressure angle** ϕ during the rise stroke, thereby decreasing lateral side thrust and friction against linear guides.

Comprehensive Comparison of Cam-Follower Configurations

Taxonomy Synthesis Matrix

Cam Type & Follower Motion & Closure Type & Key Technical Advantage
Radial Disc & Translating Radial & Spring Force & Compact structure, ease of manufacturing
Radial Disc & Translating Offset & Spring Force & Reduced peak side thrust during rise
Cylindrical Barrel & Translating Axial & Form Groove & Positive bilateral constraint in small footprint
Conjugate Pair & Oscillating Arm & Geometric Dual & Zero spring force needed, high-speed capability
Wedge Cam & Reciprocating & Gravity / Spring & Simple linear-to-linear motion conversion

Selection Criteria

Selection depends on space availability, operational speed, required force transmission, and manufacturing budget constraints.

Kinematic Design Parameters, Pressure Angle, and Summary I

Crucial Kinematic Parameters

- **Pressure Angle (ϕ):** Angle between line of follower motion and normal to pitch curve.
- **Jamming Condition:** If $\phi > 30^\circ$, excessive side thrust causes guide binding/jamming.
- **Base Circle Radius (R_b):** Increasing R_b decreases pressure angle ϕ, but increases overall mechanism footprint.
- **Undercutting:** Occurs when roller radius exceeds pitch curve radius of curvature, causing loss of intended profile.

Kinematic Design Parameters, Pressure Angle, and Summary II

Lecture Summary

- Cams convert uniform rotary input into customized non-uniform output via higher-pair contact.
- Proper choice of cam geometry (radial, cylindrical, wedge) and follower type (roller, flat-faced) optimizes mechanical efficiency and service life.
- Offsetting followers and limiting maximum pressure angle ($\phi \leq 30^\circ$) are vital for smooth, jam-free operation.

Lecture Title & Overview I

Course Details

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit:** Cams and Followers
- **Lecture 11:** 2.1 Concept, Application, and Classification of Cams
- **Instructor:** Professor of Mechanical Engineering

Concept of Cam and Follower Mechanism I

A cam-follower mechanism is a higher-pair kinematic linkage consisting of a rotating or sliding member (the cam) that imparts a predefined motion to a contacting member (the follower).

Concept of Cam and Follower Mechanism II

Key Characteristics

- **Higher Pair Joint:** Characterized by point or line contact between the driver (cam) and follower, enabling complex, non-uniform output motions that are difficult to achieve with lower pairs.
- **Three Link System:** The mechanism consists of the frame (supporting guide), the cam (input link rotating at uniform velocity), and the follower (output link executing translation or oscillation).
- **Direct Contact Transmission:** Power and motion are transmitted through direct sliding or rolling contact, often requiring form-closure (grooves) or force-closure (springs or gravity) to maintain contact.

Kinematic Principles and Joint Classification I

Analysis of the contact mechanics, degree of freedom (DoF), and motion constraints in cam mechanisms.

Kinematic Principles and Joint Classification II

Kinematic Parameters

- **Mobility Analysis:** According to Kutzbach's criterion for planar mechanisms, $F = 3(N-1) - 2P_1 - P_2$. With $N = 3$ links, $P_1 = 2$ lower pairs (cam pivot and follower guide), and $P_2 = 1$ higher pair (contact surface), the mechanism has $F = 1$ degree of freedom.
- **Constraint Types:** Form closure uses structural constraints (e.g., grooves) where the follower is trapped within a slot. Force closure relies on external forces (compression springs or gravity) to prevent separation.
- **Pressure Angle (α):** The angle between the direction of the follower's motion and the normal to the pitch curve at the contact point. To prevent jamming and high lateral forces, α should be kept below 30° for translating followers.

Diagram: Basic Radial Cam with Knife-Edge Follower I

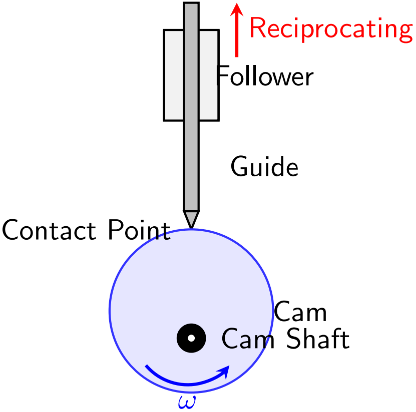


Diagram: Basic Radial Cam with Knife-Edge Follower II

- **Point Contact:** The sharp knife-edge follower makes point contact with the cam profile, permitting highly accurate profile tracing.
- **Reciprocating Path:** The follower moves linearly within the fixed guide, directly driven by the rotating radial cam profile.
- **Wear Limitations:** High contact stress at the tip causes rapid wear, making knife-edge configurations impractical for high-load or high-speed industrial operations.

Applications of Cam Mechanisms in Engineering I

Cams are widely deployed in mechanical engineering where precise, highly synchronized, and non-uniform displacement profiles are mandatory.

Applications of Cam Mechanisms in Engineering II

Key Application Areas

- **Internal Combustion Engines:** Overhead camshafts actuate intake and exhaust valves via rocker arms, ensuring exact timing relative to the piston position.
- **Automatic Manufacturing Machinery:** Disc and cylindrical cams control indexing tables, product feeders, and pick-and-place movements.
- **Textile Weaving Machines:** Complex groove cams control the precise reciprocating motion of shuttles and needles to generate patterns.
- **Printing and Packaging Equipment:** Synchronized paper feeders, cutters, and folding grippers rely on conjugate and plate cams.

Classification of Cams by Shape and Geometry I

Cams are classified by their physical geometry and the orientation of the follower's line of action relative to the cam axis.

Classification of Cams by Shape and Geometry II

Cam Geometries

- **Radial (Plate) Cams:** The cam is a flat plate with a profiled perimeter. The follower moves in a plane perpendicular to the camshaft axis.
- **Cylindrical (Barrel) Cams:** The cam has a groove machined on its outer cylindrical surface. The follower translates or oscillates parallel to the cylinder axis.
- **Wedge Cams:** A translating wedge-shaped cam slides horizontally to lift the follower vertically, ideal for low-speed adjusting mechanisms.
- **Conjugate (Double-Disk) Cams:** Comprise two cams on the same shaft acting on a common follower to maintain positive contact without springs, eliminating backlash.

Classification of Followers by Contact Type I

Follower contact tip geometry dictates the friction type, stress concentrations, and wear rates of the system.

Classification of Followers by Contact Type II

Contact Geometries

- **Knife-Edge Follower:** Simplest geometry with sharp point contact. High contact stress leads to high wear; used mostly in low-speed instrument controls.
- **Roller Follower:** Incorporates a rolling element at the contact tip. Converts sliding friction into rolling friction, minimizing wear. Most common in industrial engines.
- **Flat-Faced Follower:** Features a flat circular contacting surface. Reduces contact stress and guides lateral loads efficiently, but susceptible to misalignment.
- **Spherical-Faced (Mushroom) Follower:** Flat face modified with a spherical radius to handle misalignment and distribute stresses without high edge wear.

Classification of Followers by Motion and Path I

Kinematic classification based on the output motion of the follower and its geometric alignment with the camshaft center.

Motion and Alignment Types

- **Translating (Reciprocating) Follower:** Constrained by a guide to move in a straight line.
- **Oscillating (Rotating) Follower:** Pivots about a fixed pin axis, transforming the cam profile rotation into angular swing.
- **Radial Follower:** The line of action of the translating follower passes directly through the center of the camshaft.
- **Offset Follower:** The line of action of the follower is shifted by a distance e from the camshaft center. Offsetting reduces the side thrust on the follower guide during the rise stroke.

Diagram: Kinematics of an Offset Translating Roller Follower I

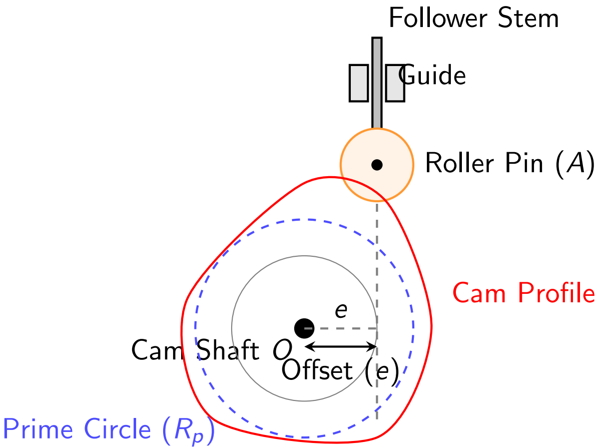


Diagram: Kinematics of an Offset Translating Roller Follower II

- **Offset Distance (e):** The offset distance shifts the follower's line of action, helping to decrease the maximum pressure angle during the critical rise stroke.
- **Prime Circle:** The circle passing through the center of the follower's roller pin when the follower is at its lowest position relative to the cam.
- **Rolling Contact:** The roller follower of radius r_f rotates freely around its pin, translating physical cam-contour variations into reciprocating displacement.

Comparison and Selection Criteria I

A comparative summary guiding mechanical engineers in selecting the optimal configuration for a given machine application.

Comparison and Selection Criteria II

Design Considerations

- **High-Speed Operation:** Roller or flat-faced followers are mandatory. Knife-edge followers must be avoided. Balanced plate profiles are required to prevent vibration.
- **Space Constraints:** Flat-faced followers allow compact designs but are susceptible to edge misalignment and require careful oil film lubrication.
- **Cost and Manufacturing:** Radial cams with flat-faced or roller followers are the most economical. Cylindrical cams require complex 3-axis CNC milling, raising cost.
- **Kinematic Security:** Form-closed (grooved) cams provide a positive mechanical drive without relying on spring forces, preventing valve float at high speed.

Table: Engineering Comparison of Follower Contact Configurations

Follower Type & **Contact Type** & **Friction** & **Wear Rate** & **Typical Applications**
Knife-Edge & Point & Sliding & Severe & Instruments, Low Speed
Roller & Line & Rolling & Very Low & IC Engines, Automation
Flat-Faced & Surface & Sliding & Low to Moderate & Valve Gear, Compact Drives
Spherical-Faced & Point / Line & Sliding & Low & High-Load Heavy Engines

Selection Summary

- **Low-Speed Precision:** Knife-edge followers give exact geometric compliance but wear rapidly.
- **High Speed & Efficiency:** Roller followers reduce friction losses and extend operating life.
- **High Load & Compactness:** Flat/spherical followers absorb heavy lateral forces efficiently.

Lecture 12 Overview & Objectives I

- **Course:** Theory of Machines and Mechanisms (GTU Course Code: DI03000141)
- **Unit 2:** Cams and Followers
- **Topic:** Kinematic Follower Motions and Displacement Diagrams
- **Primary Objectives:**
  - Understand the role of displacement diagrams (s-θ plots) in cam profile synthesis.
  - Analyze kinematic profiles: Uniform Velocity, Simple Harmonic Motion (SHM), Uniform Acceleration and Retardation (UARM), and Cycloidal Motion.
  - Master graphical drafting techniques for displacement curves.
  - Compare peak velocity, peak acceleration, and jerk characteristics for dynamic design.

Introduction to Displacement Diagrams I

- **Definition:** A displacement diagram plots the position of the follower (s) against the angular rotation of the cam (θ).
- **Coordinate Frame:**
  - **Abscissa (x-axis):** Cam rotation angle θ (0° to 360° or 0 to 2π radians).
  - **Ordinate (y-axis):** Follower linear or angular stroke/lift (h or s).
- **Cam Operation Phases:**
  - **Outstroke (θ_o):** Motion of follower from lowest position to maximum lift h.
  - **Dwell (θ_{d1}):** Follower remains stationary at full lift position.
  - **Return Stroke (θ_r):** Motion of follower back to initial position.
  - **Bottom Dwell (θ_{d2}):** Follower remains stationary at lowest position.
- **Kinematic Synthesis Foundation:** Successive time differentiation of $s(\theta)$ yields velocity $v(\theta)$, acceleration $a(\theta)$, and jerk $j(\theta)$.

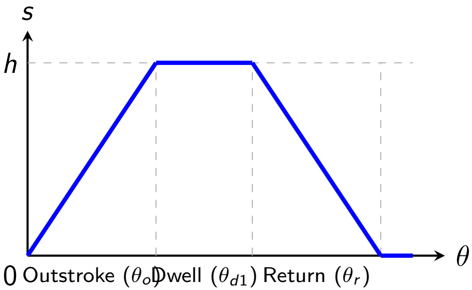
Kinematic Motion Profiles Overview I

- The choice of motion profile dictates dynamic forces, inertia shock, noise, and mechanical wear.
- **Standard Kinematic Profiles:**
  1. **Uniform Velocity Motion:** Linear position plot; constant velocity during strokes.
  2. **Simple Harmonic Motion (SHM):** Sinusoidal displacement profile; smooth moderate-speed motion.
  3. **Uniform Acceleration and Retardation (UARM):** Parabolic displacement curve; constant acceleration per half-stroke.
  4. **Cycloidal Motion:** Sinusoidally varying acceleration; continuous jerk profile for high speeds.
- **Kinematic Relations ($\omega = \frac{d\theta}{dt}$):**
  - Velocity: $v = \frac{ds}{dt} = \omega \frac{ds}{d\theta}$
  - Acceleration: $a = \frac{dv}{dt} = \omega^2 \frac{d^2s}{d\theta^2}$
  - Jerk: $j = \frac{da}{dt} = \omega^3 \frac{d^3s}{d\theta^3}$

Displacement Diagram for Uniform Velocity I

- **Governing Equation (Outstroke):**
  $s = h\left(\frac{\theta}{\theta_o}\right) \quad \text{for } 0 \leq \theta \leq \theta_o$
- **Velocity Profile:** Constant velocity $v = \frac{h\omega}{\theta_o}$.
- **Infinite Acceleration & Mechanical Shock:**
  - At stroke boundaries ($\theta = 0, \theta_o$), velocity jumps instantaneously from 0 to v.
  - Discontinuity in velocity yields theoretical acceleration $a = \pm\infty$ and Jerk $j = \pm\infty$.
  - Causes extreme inertia forces, component hammering, and high wear.
- **Modified Uniform Velocity:** Corners are rounded with small radius arcs to cap acceleration; suitable only for low operating speeds ($N < 150$ RPM).

Displacement Diagram for Uniform Velocity II



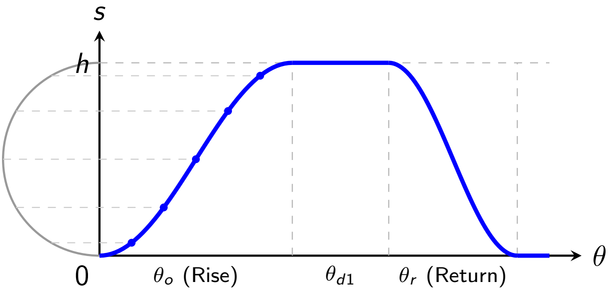
Simple Harmonic Motion (SHM) Profile I

- **Governing Outstroke Equations ($0 \le \theta \le \theta_o$):**
  - Displacement: $s = \frac{h}{2}\left[1 - \cos\left(\frac{\pi\theta}{\theta_o}\right)\right]$
  - Velocity: $v = \frac{ds}{dt} = \frac{\pi\omega h}{2\theta_o} \sin\left(\frac{\pi\theta}{\theta_o}\right)$
  - Acceleration: $a = \frac{dv}{dt} = \frac{\pi^2\omega^2 h}{2\theta_o^2} \cos\left(\frac{\pi\theta}{\theta_o}\right)$
- **Key Kinematic Features:**
  - Maximum velocity at mid-stroke ($\theta = \theta_o/2$): $v_{\max} = \frac{\pi\omega h}{2\theta_o} \approx 1.57\frac{\omega h}{\theta_o}$.
  - Maximum acceleration at start and end ($\theta = 0, \theta_o$): $a_{\max} = \frac{\pi^2\omega^2 h}{2\theta_o^2} \approx 4.93\frac{\omega^2 h}{\theta_o^2}$.
  - Acceleration is zero at mid-stroke; abrupt acceleration changes at dwell boundaries create finite jerk spikes.

Graphical Construction of SHM Displacement I

- **Step-by-Step Drafting Procedure:**
  1. Draw a semi-circle on the ordinate axis with diameter equal to follower lift h.
  2. Divide the semi-circle into n equal angular segments (e.g., 6 divisions of 30°).
  3. Divide the outstroke axis (θ_o) into the same number (n) of equal linear intervals.
  4. Project horizontal construction lines from circle divisions to intersect vertical lines drawn from corresponding angle divisions.
  5. Connect intersection points with a smooth curve.

Graphical Construction of SHM Displacement II



Uniform Acceleration and Retardation Motion (UARM) I

- **Concept:** Divides the stroke into two symmetric halves: constant acceleration followed by constant deceleration (parabolic profile).
- **Equations for Outstroke ($0 \leq \theta \leq \theta_o$):**
  - First Half ($0 \leq \theta \leq \frac{\theta_o}{2}$):
    $$s = 2h\left(\frac{\theta}{\theta_o}\right)^2, \quad v = \frac{4h\omega\theta}{\theta_o^2}, \quad a = +\frac{4h\omega^2}{\theta_o^2}$$
  - Second Half ($\frac{\theta_o}{2} \leq \theta \leq \theta_o$):
    $$s = h\left[1 - 2\left(1 - \frac{\theta}{\theta_o}\right)^2\right], \quad v = \frac{4h\omega}{\theta_o}\left(1 - \frac{\theta}{\theta_o}\right), \quad a = -\frac{4h\omega^2}{\theta_o^2}$$
- **Kinematic Characteristics:**
  - Peak velocity at mid-stroke: $v_{\max} = \frac{2h\omega}{\theta_o} = 2.00\frac{\omega h}{\theta_o}$.
  - Constant acceleration magnitude: $a_{\max} = \frac{4h\omega^2}{\theta_o^2} = 4.00\frac{\omega^2 h}{\theta_o^2}$ (lower peak acceleration than SHM).

Uniform Acceleration and Retardation Motion (UARM) II

- Instantaneous step change in acceleration at start, mid-stroke, and end causes infinite jerk impulses.

Cycloidal Motion Profile I

- **Derivation:** Generated by a point on a rolling circle of radius $r = \frac{h}{2\pi}$ along the stroke axis.
- **Complete Kinematic Equations (Outstroke):**
  - Displacement: $s = h\left[\frac{\theta}{\theta_o} - \frac{1}{2\pi}\sin\left(\frac{2\pi\theta}{\theta_o}\right)\right]$
  - Velocity: $v = \frac{h\omega}{\theta_o}\left[1 - \cos\left(\frac{2\pi\theta}{\theta_o}\right)\right]$
  - Acceleration: $a = \frac{2\pi h\omega^2}{\theta_o^2}\sin\left(\frac{2\pi\theta}{\theta_o}\right)$
- **Superior Dynamic Characteristics:**
  - Acceleration starts at zero, smoothly reaches peak $a_{\max} = \frac{2\pi h\omega^2}{\theta_o^2} \approx 6.28\frac{\omega^2 h}{\theta_o^2}$, and returns to zero.
  - Jerk remains finite throughout the entire stroke.
  - Eliminates mechanical shock and acoustic noise, making it the industry standard for ultra high-speed applications.

Comprehensive Comparison of Motion Profiles I

Table: Kinematic Comparison of Follower Motion Profiles

Motion Profile & **Max Velocity ($v_{\max}$)** & **Max Acceleration ($a_{\max}$)** & **Boundary Jerk (j)** & **Recommended Speed Range**

Uniform Velocity & $1.00 \cdot \frac{h\omega}{\theta_o}$ & ∞ & Infinite (∞) & Very low speed ($N < 150$ RPM)

Simple Harmonic (SHM) & $1.57 \cdot \frac{h\omega}{\theta_o}$ & $4.93 \cdot \frac{h\omega^2}{\theta_o^2}$ & Infinite (∞) & Moderate speed ($150 - 300$ RPM)

UARM (Parabolic) & $2.00 \cdot \frac{h\omega}{\theta_o}$ & $4.00 \cdot \frac{h\omega^2}{\theta_o^2}$ & Infinite (∞ jumps) & Medium speed ($300 - 1000$ RPM)

Cycloidal Motion & $2.00 \cdot \frac{h\omega}{\theta_o}$ & $6.28 \cdot \frac{h\omega^2}{\theta_o^2}$ & Finite & High speed ($N > 1000$ RPM)

- **Trade-off Analysis:**
  - **UARM vs. SHM:** UARM provides lower maximum acceleration (4.00 vs 4.93), reducing inertia force amplitude.
  - **Cycloidal Motion Advantage:** Although peak acceleration is highest (6.28), jerk is continuous and finite, avoiding mechanical vibration.

Design Guidelines & Operating Speed Selection I

- **Practical Selection Rules:**
  - **Low Speeds ($N < 150$ RPM):** Uniform velocity with rounded corners is acceptable.
  - **Moderate Speeds ($150 - 300$ RPM):** SHM is widely chosen due to simple geometric construction and ease of tooling.
  - **High Speeds ($300 - 1000$ RPM):** UARM is preferred over SHM to minimize inertia force peak magnitude.
  - **Ultra High Speeds ($N > 1000$ RPM):** Cycloidal motion is mandatory to eliminate infinite jerk and prevent valve floating or spring surge.
- **Optimization Strategies:**
  - Minimize stroke h and maximize outstroke angle θ_o to decrease peak velocity and acceleration.
  - Maintain high structural rigidity of the follower link to avoid elastic deformation under inertia loads.

Lecture Summary & Practice Problems I

- **Key Takeaways:**
  - Displacement diagrams $s(\theta)$ form the starting point for kinematic cam profile design.
  - Velocity, acceleration, and jerk profiles dictate mechanical performance, dynamic stress, and acoustic noise.
  - Cycloidal motion offers superior dynamic performance for high-speed automated machinery.
- **Practice Problems:**
  1. A cam operating at 600 RPM has a stroke of 40 mm over an outstroke of 90°. Calculate the maximum velocity and maximum acceleration for: (a) SHM, (b) UARM, and (c) Cycloidal motion.
  2. Explain why uniform velocity motion generates infinite acceleration at phase boundaries and how modified uniform velocity resolves this issue.
  3. Derive the expression for maximum velocity during outstroke for Simple Harmonic Motion.

Lecture 13: Follower Motions and Displacement Diagrams I

Course Information

Course: Theory of Machines and Mechanisms (DI03000141)
Unit: Cams and Followers

Lecture 13: Follower Motions and Displacement Diagrams II

Lecture Overview

- **Kinematics of Followers**: Formal definitions of displacement, velocity, acceleration, and jerk.
- **Displacement Diagram Construction**: Graphical and analytical methods for standard follower profiles.
- **Kinematic Profiles**: Detailed study of Uniform Velocity, Simple Harmonic Motion (SHM), Uniform Acceleration and Retardation Motion (UARM), and Cycloidal Motion.
- **Design Focus**: Matching mathematical motion characteristics to operational speed and dynamic requirements.

Kinematic Definitions of Follower Motion I

Follower motion parameters are analyzed as functions of cam shaft rotation angle θ (or time t, where $\theta = \omega t$ and ω is the constant angular velocity of the cam).

- **Displacement (s)**: The linear or angular position of the follower measured from its lowest position (inner dead center).
- **Velocity (v)**: The rate of change of displacement with respect to time:

$$v = \frac{ds}{dt} = \omega \cdot \frac{ds}{d\theta}$$

  Determines the hydraulic/pneumatic pressure demands and dynamic flow rates in cam-driven systems.

Kinematic Definitions of Follower Motion II

- **Acceleration (a)**: The rate of change of velocity with respect to time:
$$a = \frac{dv}{dt} = \omega^2 \cdot \frac{d^2 s}{d\theta^2}$$
Directly determines the magnitude of inertia forces ($F = m \cdot a$) acting on the cam mechanism and follower linkage.
- **Jerk (j)**: The rate of change of acceleration with respect to time:
$$j = \frac{da}{dt} = \omega^3 \cdot \frac{d^3 s}{d\theta^3}$$
Theoretical infinite jerk causes severe impact loading, dynamic shock, high-frequency noise, vibration, and rapid surface wear (spalling).

Types of Standard Follower Motions I

Cam profile design mandates choosing an appropriate mathematical motion function to control boundary accelerations and force transitions.

Uniform Velocity Motion Linear relation between displacement and angle. Constant velocity throughout the stroke; generates theoretical infinite acceleration and jerk at stroke ends.

Modified Uniform Velocity Ends of the constant velocity stroke are rounded with parabolic arcs to mitigate acceleration spikes.

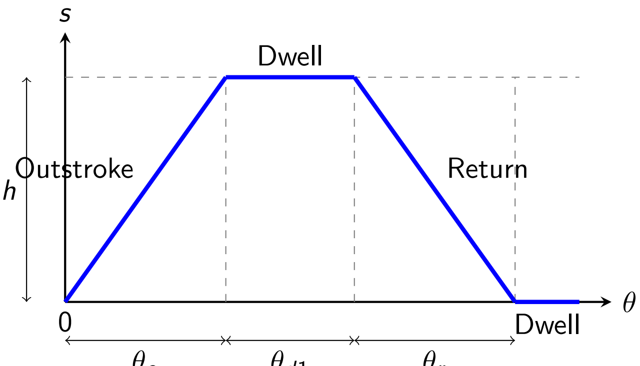
Simple Harmonic Motion (SHM) Modelled using trigonometric sine/cosine functions. Smooth velocity transition, but features step discontinuities in acceleration at stroke terminals.

Types of Standard Follower Motions II

Uniform Acceleration and Retardation (UARM) Parabolic displacement profile. Minimizes peak acceleration for a given stroke, but step changes in acceleration yield infinite jerk at start, midpoint, and end.

Cycloidal Motion Generated by a point on a rolling circle. Zero acceleration at stroke boundaries and a continuous acceleration profile with finite jerk everywhere; ideal for high-speed mechanisms.

Uniform Velocity Motion Diagram I



Displacement Diagram: Constant Velocity Profile

- **Displacement**: Increases linearly with rotation angle during outstroke: $s = h\left(\frac{\theta}{\theta_o}\right)$.

Uniform Velocity Motion Diagram II

- **Velocity**: Constant value throughout stroke: $v = \frac{h\omega}{\theta_o}$.
- **Acceleration & Jerk**: Abrupt velocity jumps from 0 to v at $\theta = 0$ and from v to 0 at $\theta = \theta_o$ result in $a = \pm\infty$ and $j = \pm\infty$.
- **Application Limit**: Restricted strictly to low-speed manual or light machine tool feeds (< 50 RPM).

Simple Harmonic Motion (SHM)

Simple Harmonic Motion uses cosine/sine relations to provide a smooth velocity profile.

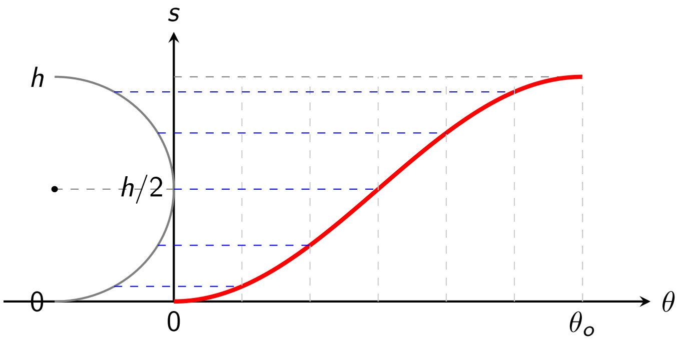
Governing Equations for Outstroke ($0 \leq \theta \leq \theta_o$)

- **Displacement**:
$$s = \frac{h}{2}\left[1 - \cos\left(\frac{\pi\theta}{\theta_o}\right)\right]$$
- **Velocity**:
$$v = \frac{ds}{dt} = \frac{\pi h\omega}{2\theta_o}\sin\left(\frac{\pi\theta}{\theta_o}\right)$$
Maximum velocity occurs at mid-stroke ($\theta = \theta_o/2$): $v_{\max} = \frac{\pi h\omega}{2\theta_o} \approx 1.57\frac{h\omega}{\theta_o}$.
- **Acceleration**:
$$a = \frac{dv}{dt} = \frac{\pi^2 h\omega^2}{2\theta_o^2}\cos\left(\frac{\pi\theta}{\theta_o}\right)$$
Maximum acceleration occurs at the stroke boundaries ($\theta = 0, \theta_o$): $a_{\max} = \frac{\pi^2 h\omega^2}{2\theta_o^2} \approx 4.93\frac{h\omega^2}{\theta_o^2}$.

Kinematic Characteristic

At stroke ends, acceleration instantly changes value from 0 to $a_{\max}$, causing infinite jerk pulses at $\theta = 0$ and $\theta = \theta_o$.

Simple Harmonic Motion Diagram I



SHM Semi-Circle Construction Method

Simple Harmonic Motion Diagram II

Graphical Construction Steps

1. Draw a semi-circle of diameter equal to total stroke lift h.
2. Divide the semi-circle circumference into an even number of equal angular divisions (e.g., 6 parts).
3. Divide the cam outstroke angle θ_o along the horizontal axis into the identical number of equal divisions.
4. Project horizontal lines from semi-circle points to intersect corresponding vertical angle gridlines; join points to form the smooth harmonic displacement curve.

Uniform Acceleration and Retardation (UARM) I

Also referred to as Parabolic Motion, UARM splits the outstroke into two equal parabolic phases.

Phase 1: Constant Acceleration ($0 \leq \theta \leq \theta_o/2$)

- **Displacement**: $s = 2h\left(\frac{\theta}{\theta_o}\right)^2$
- **Velocity**: $v = \frac{4h\omega\theta}{\theta_o^2}$
- **Acceleration**: $a = +\frac{4h\omega^2}{\theta_o^2}$ (Constant)

Uniform Acceleration and Retardation (UARM) II

Phase 2: Constant Retardation ($\theta_o/2 \leq \theta \leq \theta_o$)

- **Displacement**: $s = h - 2h\left(1 - \frac{\theta}{\theta_o}\right)^2$
- **Velocity**: $v = \frac{4h\omega}{\theta_o}\left(1 - \frac{\theta}{\theta_o}\right)$
- **Acceleration**: $a = -\frac{4h\omega^2}{\theta_o^2}$ (Constant)

- **Peak Velocity**: Occurs at mid-stroke ($\theta = \theta_o/2$): $v_{\max} = \frac{2h\omega}{\theta_o}$.
- **Peak Acceleration**: $a_{\max} = \frac{4h\omega^2}{\theta_o^2}$ (Lowest peak acceleration among non-linear profiles).
- **Drawback**: Step changes between $+a$ and $-a$ produce infinite jerk at $\theta = 0, \theta_o/2, \theta_o$.

Cycloidal Motion: The High-Speed Standard

Cycloidal motion is generated mathematically by the trace of a point on a circle rolling along a straight vertical axis.

Kinematic Equations for Outstroke ($0 \leq \theta \leq \theta_o$)

- **Displacement**:

$$s = h\left[\frac{\theta}{\theta_o} - \frac{1}{2\pi}\sin\left(\frac{2\pi\theta}{\theta_o}\right)\right]$$

- **Velocity**:

$$v = \frac{h\omega}{\theta_o}\left[1 - \cos\left(\frac{2\pi\theta}{\theta_o}\right)\right]$$

  Maximum velocity occurs at mid-stroke ($\theta = \theta_o/2$): $v_{\max} = \frac{2h\omega}{\theta_o}$.

- **Acceleration**:

$$a = \frac{2\pi h\omega^2}{\theta_o^2}\sin\left(\frac{2\pi\theta}{\theta_o}\right)$$

  Maximum acceleration occurs at $\theta = \theta_o/4$: $a_{\max} = \frac{2\pi h\omega^2}{\theta_o^2} \approx 6.28\frac{h\omega^2}{\theta_o^2}$.

- **Jerk**:

$$j = \frac{4\pi^2 h\omega^3}{\theta_o^3}\cos\left(\frac{2\pi\theta}{\theta_o}\right)$$

Key Advantage

Acceleration starts and ends at exactly zero ($a(0) = a(\theta_o) = 0$). Jerk remains finite ($j_{\max} = \frac{4\pi^2 h\omega^3}{\theta_o^3}$) across the entire rotation, eliminating impact dynamic shock completely.

Kinematic Comparison of Follower Motions I

Table: Peak Parameter Values for Lift h, Outstroke Angle θ_o, and Speed ω

Motion Profile	Max Velocity ($v_{\max}$)	Max Accel. ($a_{\max}$)	Max Jerk ($j_{\max}$)	Jerk Discontinuity
Uniform Velocity	$1.00\,\frac{h\omega}{\theta_o}$	∞	∞	Infinite at Terminals
Simple Harmonic	$1.57\,\frac{h\omega}{\theta_o}$	$4.93\,\frac{h\omega^2}{\theta_o^2}$	∞	Infinite at Terminals
Parabolic (UARM)	$2.00\,\frac{h\omega}{\theta_o}$	$4.00\,\frac{h\omega^2}{\theta_o^2}$	∞	Infinite at $0, \frac{\theta_o}{2}, \theta_o$
Cycloidal	$2.00\,\frac{h\omega}{\theta_o}$	$6.28\,\frac{h\omega^2}{\theta_o^2}$	$39.48\,\frac{h\omega^3}{\theta_o^3}$	Completely Finite

Comparative Insights

- **Peak Acceleration**: Parabolic motion yields the minimum possible peak acceleration (4.00), but causes severe jerk spikes.
- **High-Speed Suitability**: Cycloidal motion incurs slightly higher peak acceleration (6.28), but its continuous finite jerk profile prevents valve bounce and dynamic resonance.

Dynamic Selection Criteria for Cam Profiles I

Selecting an appropriate follower motion profile depends primarily on operating speed and dynamic force constraints.

Speed Regime Guidelines

- **Low Speed (<150 RPM)**: Uniform velocity or modified uniform velocity can be safely employed because inertial forces ($F = m \cdot a$) are negligible.
- **Medium Speed ($150 - 600$ RPM)**: Simple Harmonic Motion (SHM) or Parabolic Motion (UARM) is preferred due to ease of geometric layout and manufacturing simplicity.
- **High Speed (> 600 RPM)**: Cycloidal motion is mandatory. Essential for internal combustion engine valve trains and high-speed packaging automation to avoid infinite jerk spikes and resonance.

Dynamic Selection Criteria for Cam Profiles II

Manufacturing Consideration

High-speed profiles (like Cycloidal) require precision CNC profile grinding. Minor surface profile errors can introduce secondary acceleration spikes, nullifying theoretical finite-jerk benefits.

Lecture 14: Follower Motions & Displacement Diagrams – Overview I

Course Context & Objectives

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit 2**: Cam Mechanisms
- **Lecture Goal**: Systematic analysis and synthesis of follower motion profiles and displacement diagrams.

Lecture 14: Follower Motions & Displacement Diagrams – Overview II

Key Topics Covered

- Fundamentals of follower displacement diagrams ($s - \theta$ plots) and kinematic derivatives (v, a, j).
- Analysis of **Uniform Velocity Motion** and boundary discontinuity problems.
- Kinematic equations and geometric construction of **Simple Harmonic Motion (SHM)**.
- **Parabolic Motion** (Constant Acceleration and Deceleration).
- **Cycloidal Motion**: High-speed characteristics and zero-acceleration boundary conditions.
- Comparative evaluation and profile selection criteria based on operating speed and dynamic forces.

Fundamentals of Follower Displacement Diagrams I

Definition and Coordinate Representation

A **displacement diagram** represents the linear or angular displacement of the follower (s) plotted against the angular rotation of the cam ($\theta \in [0, 2\pi]$ or 0° to 360°).

- **Abscissa (x-axis)**: Cam angle θ (or time $t = \theta/\omega$, where ω is constant angular velocity).
- **Ordinate (y-axis)**: Follower displacement s (or stroke y).

Fundamentals of Follower Displacement Diagrams II

Phase Classification

A standard motion cycle consists of four distinct sequential phases:

1. **Rise / Outstroke (β_1 or θ_o)**: Follower moves from minimum to maximum position.
2. **Dwell (θ_{d1})**: Follower remains stationary at maximum lift h.
3. **Return / Instroke (β_2 or θ_r)**: Follower returns from maximum lift to base position.
4. **Dwell (θ_{d2})**: Follower remains stationary at base circle radius.

Fundamentals of Follower Displacement Diagrams III

Kinematic Derivatives

- **Velocity** (v): First derivative of displacement with respect to time.

$$v = \frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{ds}{d\theta}$$

- **Acceleration** (a): Second derivative of displacement with respect to time.

$$a = \frac{dv}{dt} = \omega^2 \frac{d^2 s}{d\theta^2}$$

- **Jerk** (j): Rate of change of acceleration (da/dt), governing mechanical shocks and vibration.

Uniform Velocity (Constant Velocity) Motion I

Kinematic Equations (Rise Phase: $0 \leq \theta \leq \beta$)

- **Displacement**:

$$s(\theta) = h\left(\frac{\theta}{\beta}\right)$$

- **Velocity**:

$$v(\theta) = \frac{ds}{dt} = \frac{h\omega}{\beta} = \text{constant}$$

- **Acceleration**:

$$a(\theta) = \frac{dv}{dt} = 0 \quad (\text{for } 0 < \theta < \beta)$$

Uniform Velocity (Constant Velocity) Motion II

Boundary Discontinuity & Theoretical Impulses

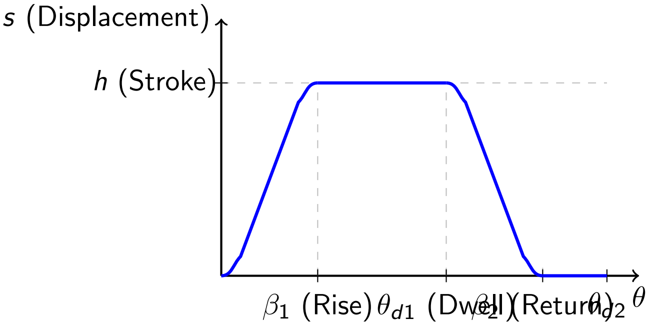
- At transition points ($\theta = 0$ and $\theta = \beta$), follower velocity changes instantaneously from 0 to $\frac{h\omega}{\beta}$ and back to 0.
- Step change in velocity causes **infinite acceleration** ($a \to \infty$) at the boundaries.
- Infinite acceleration induces infinite inertia forces ($F_{\text{inertia}} = m \cdot a$), generating severe shock loads, noise, and rapid surface fatigue.
- **Conclusion**: Pure uniform velocity profile is unsuitable for machine operations, except at extremely low manual speeds.

Displacement Diagram: Uniform Velocity with Modified Ends I

Modification Principle

To prevent infinite acceleration spikes, the straight-line displacement profile is modified at both boundary ends using parabolic arcs, rounding off sharp corners to provide finite, manageable acceleration.

Displacement Diagram: Uniform Velocity with Modified Ends II



Displacement Diagram: Uniform Velocity with Modified Ends III

Applications

Modified uniform velocity is applied in low-to-medium speed industrial applications like automatic screw machines and conveyor indexing devices.

Simple Harmonic Motion (SHM): Mathematical Formulation

Kinematic Equations (Rise Phase: $0 \leq \theta \leq \beta$)

Simple Harmonic Motion represents a smooth displacement profile derived from trigonometric cosine modulation:

- **Displacement**:

$$s(\theta) = \frac{h}{2}\left[1 - \cos\left(\frac{\pi\theta}{\beta}\right)\right]$$

- **Velocity**:

$$v(\theta) = \frac{ds}{dt} = \frac{\pi\omega h}{2\beta}\sin\left(\frac{\pi\theta}{\beta}\right)$$

- **Maximum Velocity** (occurs at midpoint $\theta = \beta/2$):

$$v_{\max} = \frac{\pi\omega h}{2\beta} \approx 1.5708\frac{h\omega}{\beta}$$

- **Acceleration**:

$$a(\theta) = \frac{dv}{dt} = \frac{\pi^2\omega^2 h}{2\beta^2}\cos\left(\frac{\pi\theta}{\beta}\right)$$

- **Maximum Acceleration** (occurs at phase ends $\theta = 0$ and $\theta = \beta$):

$$a_{\max} = \pm\frac{\pi^2\omega^2 h}{2\beta^2} \approx \pm 4.9348\frac{h\omega^2}{\beta^2}$$

Kinematic Characteristics & Limits

- Smooth continuous displacement curve with zero initial and final velocity.
- Non-zero acceleration at boundary transitions ($\theta = 0, \beta$) causes a sudden jump when connected to dwells, resulting in infinite jerk ($j \to \infty$).
- Preferred for low-to-moderate speed machinery.

Geometric Construction of SHM Displacement Curve

Graphical Drafting Procedure

To manually lay out the SHM displacement diagram:

1. Draw a semicircle with a diameter equal to the follower stroke h on the ordinate axis.
2. Divide the semicircle into an even number of equal angular segments (typically 6 or 8 parts).
3. Divide the cam rise angle β along the abscissa into the identical number of equal linear divisions.
4. Project horizontal construction lines from the semicircle division points.
5. Project vertical lines from the corresponding cam angle divisions on the abscissa.
6. Plot the intersection points and draw a smooth continuous curve through them.

Engineering Relevance

This construction guarantees exact correspondence with analytical trigonometric cosine equations during manual drafting or CAD layout verification.

Parabolic Motion (Constant Acceleration and Deceleration) I

Kinematic Formulation (2-3-4 Polynomial Motion)

Divided into two equal symmetric segments joined tangentially at the midpoint ($\theta = \beta/2$):

- **First Half: Constant Acceleration ($0 \leq \theta \leq \beta/2$)**

$$s(\theta) = 2h\left(\frac{\theta}{\beta}\right)^2, \quad v(\theta) = \frac{4h\omega\theta}{\beta^2}, \quad a(\theta) = +\frac{4h\omega^2}{\beta^2}$$

- **Second Half: Constant Deceleration ($\beta/2 \leq \theta \leq \beta$)**

$$s(\theta) = h - 2h\left(1 - \frac{\theta}{\beta}\right)^2, \quad v(\theta) = \frac{4h\omega}{\beta}\left(1 - \frac{\theta}{\beta}\right), \quad a(\theta) = -\frac{4h\omega^2}{\beta^2}$$

Parabolic Motion (Constant Acceleration and Deceleration) II

Key Kinematic Parameters

- **Peak Velocity** (at $\theta = \beta/2$): $v_{\text{max}} = 2.00\frac{h\omega}{\beta}$
- **Peak Acceleration**: $a_{\text{max}} = \pm\frac{4h\omega^2}{\beta^2} = 4.00\frac{h\omega^2}{\beta^2}$
- Provides the lowest possible peak acceleration value among standard profiles.
- Finite step changes in acceleration at $\theta = 0, \beta/2, \beta$ cause infinite jerk impulses.

Cycloidal Motion: The High-Speed Standard

Geometric Definition

Generated by tracing a point on a circle of radius $R = \frac{h}{2\pi}$ rolling without slipping along the displacement axis.

Kinematic Equations (Rise Phase: $0 \leq \theta \leq \beta$)

- **Displacement**:

$$s(\theta) = h\left[\frac{\theta}{\beta} - \frac{1}{2\pi}\sin\left(\frac{2\pi\theta}{\beta}\right)\right]$$

- **Velocity**:

$$v(\theta) = \frac{ds}{dt} = \frac{h\omega}{\beta}\left[1 - \cos\left(\frac{2\pi\theta}{\beta}\right)\right]$$

- **Maximum Velocity** (at $\theta = \beta/2$):

$$v_{\max} = \frac{2h\omega}{\beta}$$

- **Acceleration**:

$$a(\theta) = \frac{dv}{dt} = \frac{2\pi h\omega^2}{\beta^2}\sin\left(\frac{2\pi\theta}{\beta}\right)$$

- **Maximum Acceleration** (at $\theta = \beta/4$ and $\theta = 3\beta/4$):

$$a_{\max} = \pm\frac{2\pi h\omega^2}{\beta^2} \approx \pm 6.2832\frac{h\omega^2}{\beta^2}$$

High-Speed Suitability

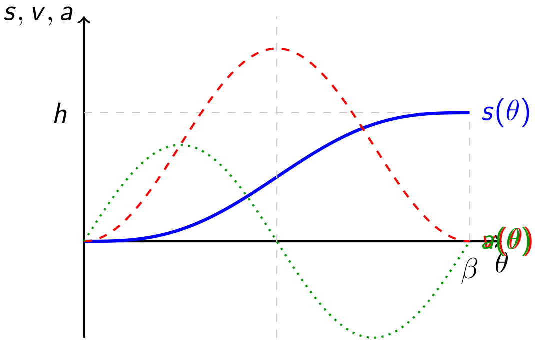
At phase boundaries ($\theta = 0$ and $\theta = \beta$), acceleration is exactly zero ($a(0) = a(\beta) = 0$). This eliminates acceleration jumps, ensuring finite jerk and smooth transition into dwell phases.

Displacement, Velocity, and Acceleration for Cycloidal Motion I

Kinematic Curves Overview

The diagram below illustrates the continuous relationship between displacement $s(\theta)$, velocity $v(\theta)$, and acceleration $a(\theta)$ across the rise phase.

Displacement, Velocity, and Acceleration for Cycloidal Motion II



Displacement, Velocity, and Acceleration for Cycloidal Motion III

Kinematic Observations

- Displacement s has horizontal tangents at both boundary limits.
- Velocity v follows a smooth sine-loop curve with peak at $\theta = \beta/2$.
- Acceleration a starts at zero, peaks positively at $\beta/4$, passes zero at $\beta/2$, peaks negatively at $3\beta/4$, and returns smoothly to zero at β.

Quantitative Comparison of Follower Motion Profiles I

Comparative Kinematic Analysis Table

For given values of stroke h, rise angle β, and cam speed ω:

Motion Profile	**Max Velocity (v_{max})**	**Max Acceleration (a_{max})**	**Boundary a**	**Jerk (j)**	**Primary Application**
Uniform Velocity	$1.00\frac{h\omega}{\beta}$	∞	Discontinuous	∞	Very low speed, manual devices
SHM	$1.57\frac{h\omega}{\beta}$	$4.93\frac{h\omega^2}{\beta^2}$	Non-zero	∞	Medium speed ($<$ 600 RPM)
Parabolic Motion	$2.00\frac{h\omega}{\beta}$	$4.00\frac{h\omega^2}{\beta^2}$	Step change	∞	Medium speed (Min Peak a)
Cycloidal Motion	$2.00\frac{h\omega}{\beta}$	$6.28\frac{h\omega^2}{\beta^2}$	Zero	Finite	High speed ($>$ 1000 RPM)

Key Trade-offs

- **Parabolic Motion** yields the lowest maximum acceleration ($4.00\frac{h\omega^2}{\beta^2}$), reducing peak inertia forces.
- **Cycloidal Motion** has higher peak acceleration ($6.28\frac{h\omega^2}{\beta^2}$), but eliminates jerk spikes at boundary transitions.

Practical Selection Guide for Cam-Follower Motions I

Selection Guidelines Based on Operating Speed

- **Low Speed (<300 RPM)**: Modified Uniform Velocity or Parabolic Motion. Peak dynamic forces are minimal.
- **Medium Speed ($300-1000$ RPM)**: Simple Harmonic Motion (SHM) or Parabolic Motion. Moderate inertia forces; spring return stiffness must overcome follower mass.
- **High Speed (>1000 RPM)**: Cycloidal Motion or Advanced Polynomial Profiles ($3-4-5$ or $4-5-6-7$). Requires smooth acceleration continuity to prevent high-frequency vibration, noise, and floating followers.

GTU Examination Focus Areas

- Derive maximum velocity and acceleration expressions for SHM and Cycloidal motions.
- Draw displacement, velocity, and acceleration diagrams for specified cam motion cycles.
- Graphical construction of displacement diagrams using semicircle division (SHM) or tangent divisions (Parabolic).

Lecture 15: 2.3 Drawing of Cam Profiles I

Course: Theory of Machines and Mechanisms (DI03000141)
Unit: Cams and Followers
Topic: 2.3 Drawing of Cam Profiles

Lecture Objectives

- Understand essential cam nomenclature required for graphical construction.
- Master construction of displacement diagrams for various follower motion patterns.
- Graphically construct cam profiles for radial knife-edge, roller, offset, and flat-faced followers.
- Evaluate critical design parameters such as pressure angle (ϕ) and prevent undercutting.

Nomenclature of Cam Profiles I

- **Base Circle (r_b):** The smallest circle tangent to the cam profile, centered on the cam rotation axis. It determines the overall physical size of the cam mechanism.
- **Trace Point:** A theoretical reference point on the follower used to generate the pitch curve.
  - Knife-edge follower: Pointed contact tip.
  - Roller follower: Center of the roller pin.
  - Flat-faced follower: Intersection point of follower axis and flat contact surface.
- **Pitch Curve:** Path generated by the trace point as the follower moves relative to the stationary cam.
- **Prime Circle (r_p):** The smallest circle drawn from the cam center tangent to the pitch curve.
  - Knife-edge follower: $r_p = r_b$.
  - Roller follower: $r_p = r_b + r_r$ (where r_r is the roller radius).

- **Pressure Angle (ϕ):** The angle between the line of follower movement and the normal to the pitch curve at any point.

The Displacement Diagram I

Definition & Purpose

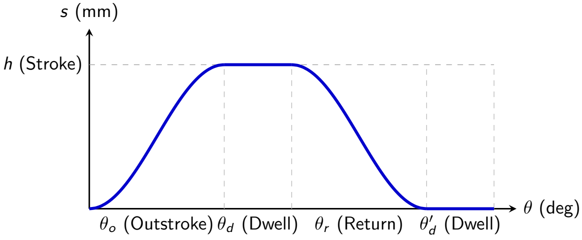
The displacement diagram plots follower lift s (vertical axis) against cam rotation angle θ (horizontal axis) over one complete revolution (360°).

- **Horizontal Axis (θ):** Divided into operational phase angles:
  - Outstroke (θ_o): Follower rises from bottom position to full stroke h.
  - Dwell (θ_d): Follower remains stationary at full stroke height h.
  - Return stroke (θ_r): Follower falls back to initial position.
  - Dwell (θ'_d): Follower remains at rest at minimum radius.
- **Vertical Axis (s):** Represents total follower stroke or lift height (h).

The Displacement Diagram II

- **Motion Curves:** Simple Harmonic Motion (SHM), Uniform Velocity, Uniform Acceleration and Retardation (UARM), and Cycloidal Motion.
- Subdivided into equal angular intervals (*e.g.*, 6 or 8 equal parts per phase) to transfer points to the cam profile layout.

Displacement Diagram Construction



- Divide outstroke θ_o and return stroke θ_r into equal angular intervals $(0, 1, 2, 3, \dots)$.
- Project displacement values $1 - 1', 2 - 2', 3 - 3'$ from the diagram onto the radial/tangent construction lines of the cam layout.

Construction for Radial Knife-Edge Follower

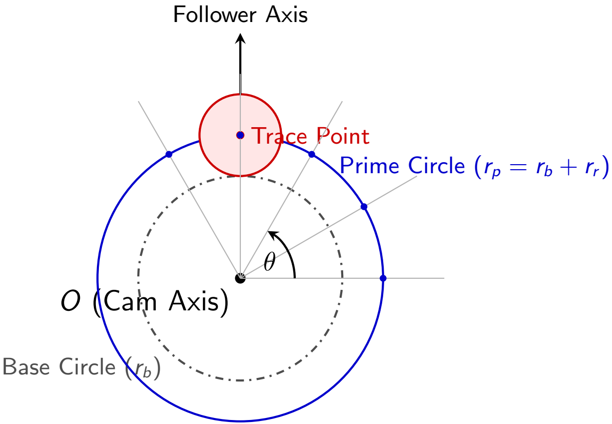
Procedure for Radial Knife-Edge Follower

1. **Draw Base Circle:** Draw circle with radius r_b centered at cam axis O.
2. **Draw Follower Axis:** Pass vertically through O, intersecting the base circle at trace point A.
3. **Layout Cam Angles:** Mark phase angles θ_o, θ_d, θ_r, θ'_d around O in direction opposite to cam rotation (ω).
4. **Divide Angles:** Divide θ_o and θ_r into the exact same number of equal divisions as in the displacement diagram.
5. **Draw Radial Rays:** Construct radial lines from O extending through each division point on the base circle.
6. **Transfer Displacements:** Measure displacement heights along each radial line starting from the base circle outwards $(1 - 1', 2 - 2', \dots)$.
7. **Draw Cam Profile:** Join all transferred points with a smooth, continuous curve.

Construction for Radial Roller Follower

Procedure for Radial Roller Follower

1. **Base & Prime Circles:** Draw base circle r_b and prime circle $r_p = r_b + r_r$ from center O.
2. **Initial Trace Point:** Locate initial roller center A on the prime circle along the vertical axis.
3. **Angular Divisions:** Divide phase angles θ_o and θ_r on the prime circle and draw radial construction lines.
4. **Transfer Displacements:** Measure displacements from displacement diagram along radial lines, starting from the prime circle, to locate roller centers $(C_1, C_2, C_3, \dots)$.
5. **Pitch Curve:** Connect roller centers C_i with a smooth curve.
6. **Roller Circles:** Draw circles of roller radius r_r centered at each C_i.
7. **Cam Profile:** Draw a smooth curve tangent to the inner envelope of all roller circles.



- The trace point generates the **pitch curve** at radial distance $r_p + s(\theta)$.

Cam Profile Construction Geometry II

- The physical cam surface is created by constructing the inner **envelope tangent** to all roller circle positions.

Offset Follower Profile Construction I

Offset Follower Profile Construction II

Procedure when Follower Axis is Offset by Distance x

1. **Base & Offset Circles:** Draw base circle (r_b) and offset circle of radius x from cam center O.
2. **Initial Position Line:** Draw line TA tangent to the offset circle from the initial follower trace point A.
3. **Angular Divisions:** Divide phase angles on the offset circle starting from tangent point T.
4. **Construct Tangents:** At each division point $T_1, T_2, \dots$ on the offset circle, draw tangent lines to the offset circle in the direction of relative motion.
5. **Transfer Displacements:** Mark displacement distances from the base circle (or prime circle) along these tangent lines to locate trace points $1', 2', 3', \dots$
6. **Cam Profile:** Connect trace points (knife-edge) or construct roller envelope curve along the offset tangent lines.

Flat-Faced Follower Profile Construction I

Flat-Faced Follower Profile Construction II

Procedure for Translating Flat-Faced Follower

1. **Base Circle:** Draw base circle r_b centered at O.
2. **Radial Rays:** Divide phase angles on the base circle and draw radial lines $O1, O2, O3, \ldots$
3. **Mark Displacements:** Measure displacements from base circle along radial lines to locate points $1', 2', 3', \ldots$ (intersections of follower axis and flat face).
4. **Draw Flat Faces:** At each point $1', 2', 3'$, draw a straight line **perpendicular** to the respective radial line.
5. **Envelope Cam Profile:** Construct a smooth curve tangent to all flat face straight lines.
6. **Face Width Criterion:** Ensure minimum flat face length extends beyond maximum contact point distance $x = \frac{ds}{d\theta}$ to prevent contact slipping off the face.

Table: Comparison of Cam Profile Layout Features across Follower Types

Follower Type & **Reference Circle** & **Trace Point Location** & **Cam Profile Construction Method**

Knife-Edge & Base Circle (r_b) & Tip of Knife-Edge & Direct smooth curve joining transferred displacement points.
Roller & Prime Circle (r_p) & Center of Roller Pin & Inner envelope curve tangent to roller circles on pitch curve.
Offset & Base & Offset (x) & Tangent Line Point & Tangents to offset circle with displacements measured along tangents.
Flat-Faced & Base Circle (r_b) & Axis-Face Intersection & Envelope curve tangent to straight lines perpendicular to radial rays.

Practical Considerations in Cam Layout I

1. Pressure Angle (ϕ) Control

- **Definition:** Angle between follower line of motion and normal to pitch curve.
- **Consequence:** Excessive pressure angle increases side thrust, leading to guide wear, bending, and mechanism jamming.
- **Allowable Limits:** $\phi_{max} \leq 30^\circ$ for translating followers; $\phi_{max} \leq 45^\circ$ for oscillating followers.
- **Remedies:** Increase base circle radius r_b or introduce follower offset x.

Practical Considerations in Cam Layout II

2. Avoidance of Undercutting

- **Condition:** Occurs when pitch curve radius of curvature $\rho_{pitch} < r_r$.
- **Consequence:** Sharp cusp formed on cam profile; actual follower motion loses stroke requirement.
- **Remedies:** Increase base circle radius r_b, decrease roller radius r_r, or adopt motion curves with lower peak acceleration (*e.g.*, Cycloidal instead of SHM/Uniform Velocity).

Lecture 16 Overview: Drawing of Cam Profiles I

Course & Unit Information

- **Course:** Theory of Machines and Mechanisms (GTU Code: DI03000141)
- **Unit II:** Cams and Followers
- **Topic:** 2.3 Graphical Construction of Cam Profiles

Lecture 16 Overview: Drawing of Cam Profiles II

Learning Objectives

- Define critical cam geometric parameters: Base Circle, Prime Circle, Pitch Curve, and Pressure Angle.
- Apply the principle of kinematic inversion to lay out cam profiles.
- Construct profiles for translating followers: knife-edge, roller, and flat-face.
- Analyze offset follower configurations and their impact on side thrust.
- Identify the physical causes of cam undercutting and determine minimum base circle radius R_b.

Key Geometric Definitions in Cam Design I

Base Circle (R_b) The smallest circle drawn tangent to the cam profile from the center of rotation O. It determines the overall physical size of the cam.

Trace Point A theoretical reference point on the follower (e.g., knife-edge tip or roller center) used to generate the kinematic motion.

Pitch Curve Path traced by the trace point if the cam is held stationary while the follower rotates around it.

Prime Circle (R_p) The smallest circle centered at O that is tangent to the pitch curve ($R_p = R_b + r$ for a roller follower of radius r).

Pressure Angle (ϕ) The angle between the normal to the pitch curve and the direction of follower motion (line of action).

Key Geometric Definitions in Cam Design II

Design Limit on Pressure Angle

A high pressure angle ($\phi > 30^\circ$) produces excessive side thrust on the follower guide, leading to mechanical binding, increased friction, and severe wear.

The Principle of Kinematic Inversion I

Theoretical Basis

Direct graphical layout of a rotating cam against a translating follower is complex. Kinematic inversion simplifies this by holding the cam stationary and rotating the follower frame relative to the cam.

The Principle of Kinematic Inversion II

Step-by-Step Mechanism of Inversion

1. Fix the cam in space at center O.
2. Rotate the follower axis around O at an angular velocity $-\omega$ (opposite to actual cam rotation).
3. At each angular step θ_i, extend the follower by displacement y_i obtained from the displacement diagram.
4. Plot the successive positions of the follower in this inverted frame.
5. Draw the cam profile as the continuous **envelope curve** tangent to all successive follower positions.

Inline Knife-Edge Follower Construction I

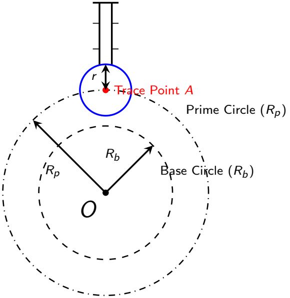
Graphical Construction Procedure

1. **Base Circle:** Draw a circle of radius R_b centered at O. Draw the initial vertical line of action through O.
2. **Angular Divisions:** Divide the total cam rotation into ascent (θ_a), dwell (θ_d), descent (θ_r), and low dwell (θ'_d). Subdivide each interval into equal angular divisions (e.g., 15° or 30°).
3. **Radial Rays:** Draw radial lines from center O through each division point on the base circle in the direction opposite to cam rotation.
4. **Displacement Transfer:** Transfer displacement values $y_1, y_2, \ldots, y_n$ from the displacement diagram along the corresponding radial lines outwards from the base circle.
5. **Profile Curve:** Join the resulting points $P_1, P_2, \ldots, P_n$ with a smooth curve to complete the cam profile.

Inline Roller Follower Construction & Geometry

Construction Steps

1. **Prime Circle:** Draw base circle (R_b) and prime circle ($R_p = R_b + r$).
2. **Pitch Curve:** Plot displacements y_i outwards from the *prime circle* along radial lines to locate roller centers A_i. Connect A_i to form the pitch curve.
3. **Roller Circles:** Draw circles of radius r centered at each A_i.
4. **Cam Profile:** Draw a smooth curve tangent to the bottom of all roller circles.



Offset Follower Construction: Kinematic Rationale I

Purpose of Offset (e)

Offsetting the follower line of action by distance e relative to cam center O reduces the pressure angle ϕ during the stroke where forces are highest (ascent), thereby reducing lateral guide friction and wear.

Offset Follower Construction: Kinematic Rationale II

Layout Steps for Offset Follower

1. Draw base circle (R_b) and offset circle of radius e centered at O.
2. Draw the initial line of action tangent to the offset circle at point T_0.
3. Divide the offset circle into angular sectors corresponding to rotation intervals.
4. At each division point T_i, draw lines tangent to the offset circle. These represent inverted lines of action.
5. Transfer displacements y_i measured along these tangent lines from the base circle (or prime circle).

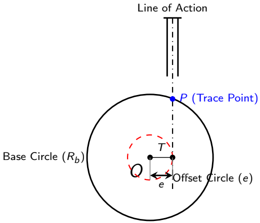
Offset Follower Geometry & Tangent Layout

Geometric Relationships

- Offset distance = e.
- Initial line of action is tangent to offset circle at point T.
- Trace point location P at zero lift:

$$OT = e, \quad OP_0 = \sqrt{R_b^2 + e^2} \text{ (or } R_p)$$

- Shifted lines of action remain tangent to offset circle throughout rotation.



Flat-Face Follower Construction & Envelope Method I

Characteristics

- Contact surface is a flat plane normal to the line of motion.
- **Pressure Angle:** $\phi = 0^\circ$ (neglecting friction), resulting in zero lateral side thrust.

Flat-Face Follower Construction & Envelope Method II

Construction Steps

1. Draw base circle (R_b) centered at O and draw radial lines at chosen angular intervals θ_i.
2. Mark displaced points P_i along radial lines such that $OP_i = R_b + y_i$.
3. At each point P_i, draw a straight line **perpendicular** to radial line OP_i (representing the flat face).
4. Draw a smooth curve tangent to all flat-face lines to generate the cam profile.

Flat-Face Follower Construction & Envelope Method III

Minimum Face Width Requirement

To prevent the contact point from running off the edge, the minimum face width W must satisfy $W > 2\left|\frac{v_{\text{max}}}{\omega}\right|$, where v_{max} is maximum follower velocity.

Cam Profile Undercutting & Base Circle Sizing I

What is Undercutting?

Undercutting occurs when the generated cam profile contains a cusp or self-intersecting loop, causing the physical cam to trim away part of the required surface and fail to produce the desired follower motion.

Mathematical Criterion for Roller Follower

- Radius of curvature of pitch curve: ρ_p.
- Radius of curvature of cam profile: $\rho_c = \rho_p - r$.
- **Undercutting Condition:** If $\rho_p < r$, then $\rho_c < 0$, producing a cusp or loop.
- **Satisfactory Condition:** $\rho_p > r$ everywhere along the profile.

Cam Profile Undercutting & Base Circle Sizing II

Remedial Measures

- Increase the Base Circle Radius (R_b), which increases ρ_p and reduces ϕ.
- Reduce the Roller Radius (r).
- Use continuous, smooth motion profiles (e.g., cycloidal motion).

Comparison of Follower Types in Cam Design I

Follower Type & **Contact** & **Side Thrust (ϕ)** & **Key Features & Design Notes**
Knife-Edge & Point & High ($\phi > 30°$) & High wear rate; limited to low-load demo applications.
Roller & Rolling & Moderate & Low friction, long life; requires prime circle layout.
Flat-Face & Line/Plane & Zero ($\phi = 0°$) & Zero side thrust; requires $W > 2\lvert v_{\max}/\omega \rvert$ to avoid runoff.
Spherical-Face & Point & Low & Eliminates edge stress concentration seen in flat-face followers.

Table: Comparative performance metrics for translating follower types.

Graphical vs. Analytical Cam Synthesis & Summary I

Graphical Method

- Intuitive visual understanding.
- Excellent for conceptual design and manual verification.
- Limited accuracy dependent on drawing scale.

Analytical Method

- Uses exact parametric equations $x(\theta), y(\theta)$.
- Directly generates CNC toolpaths (G-code).
- Essential for high-speed, high-precision cams.

Graphical vs. Analytical Cam Synthesis & Summary II

Summary of Key Design Rules

1. Maintain maximum pressure angle $\phi_{\text{max}} \leq 30^\circ$ for translating followers.
2. Ensure R_b is sufficiently large to guarantee $\rho_p > r$ and avoid undercutting.
3. Select appropriate displacement curves (SHM, Cycloidal, 3-4-5 Polynomial) based on operating speed.

Lecture 17: Friction, Uniform Pressure and Wear Theories I

Course: Theory of Machines and Mechanisms (GTU Code: DI03000141)

- Unit: Friction in Machine Elements – Pivots, Collars, and Clutches.
- Purpose: Understand axial thrust friction and surface contact behavior in rotating machinery.
- Core Theories Evaluated:
  - Uniform Pressure Theory (UPT): Applicable to un-worn, newly machined surfaces.
  - Uniform Wear Theory (UWT): Applicable to worn-in, running-in surfaces after initial operation.
- Key Topics Covered:
  - Derivation of frictional torque for flat pivots, conical pivots, and collar bearings.

Lecture 17: Friction, Uniform Pressure and Wear Theories II

- Mechanics of single-plate, multi-plate, cone, and centrifugal friction clutches.
- Comparative performance matrices and practical engineering design selection criteria.

Fundamentals of Pivot and Collar Bearings I

Pivot and collar bearings support axial thrust forces acting parallel to the rotating shaft axis.

- Pivot Bearing: Supports axial load at the terminal end of a rotating shaft (e.g., flat pivot, conical pivot).
- Collar Bearing: Supports axial thrust along an intermediate position of a continuous shaft.
- Differential Element Analysis:
  - Consider a circular surface with inner radius r_2 and outer radius r_1.
  - Elemental ring at radius r with radial thickness dr.
  - Elemental surface area: $dA = 2\pi r\, dr$.
  - Elemental axial load: $dW = p \cdot dA = p \cdot 2\pi r\, dr$, where p is normal intensity of pressure.
  - Elemental frictional force: $dF_f = \mu\, dW = \mu \cdot p \cdot 2\pi r\, dr$.
  - Elemental frictional torque about shaft axis: $dT = r \cdot dF_f = 2\pi \mu p r^2\, dr$.

Fundamentals of Pivot and Collar Bearings II

- Total Frictional Torque: $T = \int_{r_2}^{r_1} dT = 2\pi\mu \int_{r_2}^{r_1} pr^2\, dr$.

Uniform Pressure Theory (UPT) I

Uniform Pressure Theory assumes ideal, rigid contact where pressure intensity remains constant throughout.

- Key Assumption: Intensity of pressure p is uniform across the contact area ($p =$ constant).
- Surface Application: Newly constructed bearings, un-worn friction surfaces, and rigid body approximations.
- Total Axial Load (W):

$$W = \int_{r_2}^{r_1} 2\pi pr\,dr = 2\pi p \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = p\cdot\pi(r_1^2 - r_2^2) \implies p = \frac{W}{\pi(r_1^2 - r_2^2)}$$

- Total Frictional Torque (T):

$$T = 2\pi\mu p \int_{r_2}^{r_1} r^2\,dr = 2\pi\mu p \left[\frac{r^3}{3}\right]_{r_2}^{r_1} = \frac{2}{3}\pi\mu p(r_1^3 - r_2^3)$$

Uniform Pressure Theory (UPT) II

- Substituting p:

$$T = \frac{2}{3}\mu W \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) = \mu W R_m \quad \text{where } R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$

- Solid Flat Pivot ($r_2 = 0$): $R_m = \frac{2}{3}r_1 \implies T = \frac{2}{3}\mu W r_1$.

Uniform Wear Theory (UWT)

Uniform Wear Theory models worn-in surfaces where wear rate reaches a stable equilibrium state.

- Key Assumption: Wear rate is proportional to frictional work done, which depends on pressure and velocity.

$$\text{Rate of Wear} \propto p \cdot v \propto p \cdot (\omega r) \implies p \cdot r = C \text{ (constant)}$$

- Pressure Distribution: Pressure is inversely proportional to radius ($p = C/r$).
  - Maximum pressure $p_{\max}$ occurs at inner radius r_2: $C = p_{\max} \cdot r_2$.
  - Minimum pressure $p_{\min}$ occurs at outer radius r_1: $C = p_{\min} \cdot r_1$.
- Total Axial Load (W):

$$W = \int_{r_2}^{r_1} 2\pi(pr)\, dr = 2\pi C \int_{r_2}^{r_1} dr = 2\pi C(r_1 - r_2) \implies C = \frac{W}{2\pi(r_1 - r_2)}$$

- Total Frictional Torque (T):

$$T = 2\pi\mu \int_{r_2}^{r_1} (pr) r\, dr = 2\pi\mu C \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = \pi\mu C(r_1^2 - r_2^2)$$

- Substituting C:

$$T = \frac{1}{2}\mu W(r_1 + r_2) = \mu W R_m \quad \text{where } R_m = \frac{r_1 + r_2}{2}$$

- Solid Flat Pivot ($r_2 = 0$): $R_m = \frac{r_1}{2} \implies T = \frac{1}{2}\mu W r_1$.

Comparative Analysis: UPT vs. UWT I

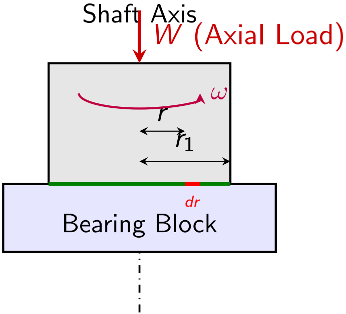
A comprehensive comparison between Uniform Pressure Theory and Uniform Wear Theory.

Parameter	**Uniform Pressure Theory (UPT)**	**Uniform Wear Theory (UWT)**
Surface Condition	New, un-worn, precisely flat contact surfaces.	Worn-in, running-in surfaces after initial use.
Pressure Profile	$p = \text{constant}$	$p \cdot r = C \implies p \propto \frac{1}{r}$
Collar Mean Radius R_m	$R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$	$R_m = \frac{r_1 + r_2}{2}$
Solid Pivot Radius R_m	$R_m = \frac{2}{3} r_1$	$R_m = \frac{1}{2} r_1$
Torque Capacity	Predicts higher torque capacity ($T_{\text{UPT}} > T_{\text{UWT}}$).	Predicts lower torque capacity ($T_{\text{UWT}} < T_{\text{UPT}}$).
Design Significance	Used for initial sizing of brand new clutches.	Preferred for conservative mechanical design.

- Note: Since $R_{m,\text{UWT}} < R_{m,\text{UPT}}$, designing friction clutches under UWT ensures safe torque capacity throughout the component's operational lifespan.

Flat Pivot Bearing Geometry I

Schematic layout of a flat pivot bearing under axial load W.



- Friction Power Loss:

$$P_{\text{loss}} = T \cdot \omega = T \cdot \left(\frac{2\pi N}{60}\right) \quad \text{[Watts]}$$

- Power loss generates heat at the contact surface, requiring suitable lubrication to prevent seizure.

Conical Pivot Bearing Mechanics I

Conical pivots guide the shaft radially while supporting axial load W.

- Semi-cone Angle: α (angle made by the cone slant face with the shaft axis).
- Contact Area Element: Slant length element $ds = \frac{dr}{\sin\alpha}$.
- Elemental Normal Load (dN):

$$dN = p \cdot (2\pi r\, ds) = \frac{p \cdot 2\pi r\, dr}{\sin\alpha}$$

- Vertical equilibrium gives axial load: $dW = dN \cdot \sin\alpha = p \cdot 2\pi r\, dr$.
- Elemental Frictional Force (dF_f):

$$dF_f = \mu\, dN = \frac{\mu \cdot p \cdot 2\pi r\, dr}{\sin\alpha}$$

Conical Pivot Bearing Mechanics II

- Total Frictional Torque (T):

$$T = \int r \, dF_f = \frac{2\pi\mu}{\sin\alpha} \int_{r_2}^{r_1} pr^2 \, dr$$

- Formulas for Conical Pivot:
  - Uniform Pressure Theory: $T = \frac{2}{3}\frac{\mu W}{\sin\alpha}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$. For solid cone ($r_2 = 0$): $T = \frac{2}{3}\frac{\mu W r_1}{\sin\alpha}$.
  - Uniform Wear Theory: $T = \frac{1}{2}\frac{\mu W}{\sin\alpha}(r_1 + r_2)$. For solid cone ($r_2 = 0$): $T = \frac{1}{2}\frac{\mu W r_1}{\sin\alpha}$.

Flat Collar and Multi-Collar Bearings I

Collar bearings distribute heavy axial thrust along multiple collar surfaces.

- Flat Collar Geometry: Shaft diameter $2r_2$, collar outer diameter $2r_1$.
- Single Collar Torque Equations:
  - Under UPT: $T = \frac{2}{3}\mu W\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$
  - Under UWT: $T = \frac{1}{2}\mu W(r_1 + r_2)$
- Multi-Collar Bearings:
  - Employed in large marine propeller shafts where axial thrust load W is extremely high.
  - If n identical collars are used, axial load per collar: $W' = \frac{W}{n}$.
  - Allowable intensity of pressure $p_{\text{allowable}}$ governs collar sizing:

$$p = \frac{W}{n\pi(r_1^2 - r_2^2)} \leq p_{\text{allowable}}$$

- Total Frictional Torque across n collars:

$$T_{\text{total}} = n \cdot (\mu W' R_m) = n \cdot \mu \left(\frac{W}{n} \right) R_m = \mu W R_m$$

- Total torque depends on total load W and mean radius R_m, independent of collar count n.

Single-Plate and Multi-Plate Friction Clutches I

Friction clutches transmit rotary motion and power between co-axial driving and driven shafts.

- Operating Principle: Axial spring force W presses friction surfaces together.
- Torque Transmission Equation:

$$T = n \cdot \mu \cdot W \cdot R_m$$

  where:
  - n: Number of effective contacting surface pairs.
  - μ: Coefficient of friction between contact linings.
  - W: Total axial engagement force.
  - R_m: Mean friction radius.
- Active Contact Surfaces (n):
  - Single-plate clutch with lining on both sides: $n = 2$.

Single-Plate and Multi-Plate Friction Clutches II

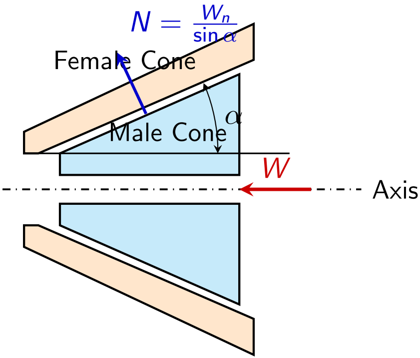
- Multi-plate clutch with n_1 driving plates and n_2 driven plates:

$$n = n_1 + n_2 - 1$$

- Selection of Theory for Design:
  - New Clutch (UPT): $R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$
  - Service / Worn Clutch (UWT): $R_m = \frac{r_1 + r_2}{2}$ (Standard design requirement).

Cone Clutch Kinematics and Forces I

Cone clutches provide higher torque capacity for a given axial force due to wedging action.



- Torque Transmission (UWT): $T = \frac{\mu W R_m}{\sin\alpha}$, where $R_m = \frac{r_1+r_2}{2}$.

Cone Clutch Kinematics and Forces II

- Axial Force for Engagement (W):

$$W = N(\sin\alpha + \mu\cos\alpha) = W_n(1 + \mu\cot\alpha)$$

- Axial Force for Disengagement (W'):

$$W' = N(\mu\cos\alpha - \sin\alpha)$$

- Self-Locking Condition: If $\tan\alpha \leq \mu$, disengagement force $W' \leq 0$ (clutch remains wedged). Semi-cone angle α is kept between 12.5° and 15° to prevent binding.

Centrifugal Clutch Principles I

Centrifugal clutches engage automatically as the driving engine speed increases.

- Operational Principle: Uses centrifugal force acting on internal shoes to press friction linings against a driven rim.
- Force Balance on Each Shoe:
  - Centrifugal force at running speed ω: $P_c = m\omega^2 R_g$, where m is shoe mass and R_g is center of gravity radius.
  - Inward spring force holding shoe: $P_s = m\omega_1^2 R_g$, where ω_1 is engagement threshold speed.
  - Net outward normal force acting on driven rim:

$$P_n = P_c - P_s = mR_g(\omega^2 - \omega_1^2)$$

- Total Frictional Torque Transmitted (T):

$$T = n \cdot \mu \cdot P_n \cdot R = n\,\mu\,m\,R_g\,R\,(\omega^2 - \omega_1^2)$$

where n is number of shoes and R is inside radius of driven rim.

Numerical Problem: Design verification of a multi-plate friction clutch.

- Given Parameters:
  - Number of driving plates $n_1 = 3$, number of driven plates $n_2 = 2$.
  - Outer radius $r_1 = 120$ mm $= 0.12$ m, inner radius $r_2 = 80$ mm $= 0.08$ m.
  - Axial spring force $W = 1.5$ kN $= 1500$ N, friction coefficient $\mu = 0.3$.
  - Shaft speed $N = 1500$ rpm.
- Step 1: Active pairs of contact surfaces:

$$n = n_1 + n_2 - 1 = 3 + 2 - 1 = 4$$

Step-by-Step Numerical Application & Summary II

- Step 2: Mean friction radius under UWT (worn-in assumption):

$$R_m = \frac{r_1 + r_2}{2} = \frac{0.12 + 0.08}{2} = 0.10 \text{ m}$$

- Step 3: Torque capacity calculation:

$$T = n\mu W R_m = 4 \times 0.3 \times 1500 \text{ N} \times 0.10 \text{ m} = 180 \text{ N} \cdot \text{m}$$

- Step 4: Power transmission capacity:

$$P = \frac{2\pi N T}{60} = \frac{2\pi \times 1500 \times 180}{60} = 28274.3 \text{ W} = 28.27 \text{ kW}$$

- Key Lesson: Uniform Wear Theory yields conservative estimates, ensuring safe power transmission over time.

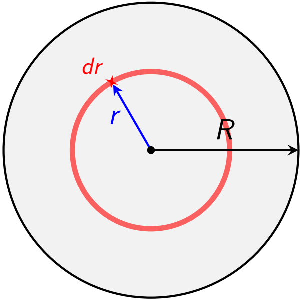
Lecture Overview & Objectives I

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit**: Bearings, Clutches, Brakes & Dynamometers
- **Core Objectives**:
  - Understand the role of friction in thrust loads on pivot and collar bearings.
  - Derive frictional torque using **Uniform Pressure Theory (UPT)**.
  - Derive frictional torque using **Uniform Wear Theory (UWT)**.
  - Compare design implications of UPT vs. UWT for clutches and bearings.
  - Extend analytical models to flat collar bearings and conical friction surfaces.
  - Quantify frictional power loss and heat dissipation requirement.

Introduction to Friction in Pivots and Collars I

- **Dual Nature of Friction in Machine Elements**:
  - *Undesirable (Bearings)*: Causes continuous mechanical power loss and thermal degradation; must be minimized.
  - *Desirable (Clutches & Brakes)*: Relies on surface friction to transmit torque or absorb kinetic energy.
- **Classification of Thrust Support Members**:
  - **Pivot Bearing**: Supports axial thrust load W at the terminus (end) of a shaft (e.g., solid flat pivot, conical pivot).
  - **Collar Bearing**: Supports axial thrust load W at any intermediate section along a shaft using an annular surface.
- **Fundamental Governing Assumption**: Contact surfaces are perfectly flat and aligned, with coefficient of friction μ remaining uniform across the interface.

Flat Pivot Bearing Analysis I



Elemental Area $dA = 2\pi r\, dr$

- Consider a solid shaft of radius R bearing an axial thrust load W.
- An elemental ring at radius r with width dr has contact area:

$$dA = 2\pi r\, dr$$

Flat Pivot Bearing Analysis II

- Normal load on elemental ring: $dN = p \cdot dA = 2\pi pr\,dr$.
- Elemental frictional force: $dF = \mu\,dN = 2\pi\mu pr\,dr$.
- Elemental frictional torque: $dT = dF \cdot r = 2\pi\mu pr^2\,dr$.

Uniform Pressure Theory (UPT) I

- **Core Hypothesis**: The intensity of pressure p is uniformly distributed over the entire contact surface ($p =$ constant).
- **Applicability**: New, unworn, precisely machined flat contact surfaces.
- **Axial Load Relation**:

$$W = \int_0^R p \cdot 2\pi r \, dr = p \cdot \pi R^2 \implies p = \frac{W}{\pi R^2}$$

- **Total Frictional Torque Derivation**:

$$T = \int_0^R dT = \int_0^R 2\pi \mu p r^2 \, dr = 2\pi \mu p \left[\frac{r^3}{3}\right]_0^R = \frac{2}{3}\pi \mu p R^3$$

Uniform Pressure Theory (UPT) II

- Substituting $p = \frac{W}{\pi R^2}$ gives:

$$T = \frac{2}{3}\mu WR$$

- **Equivalent Mean Friction Radius**: $R_m = \frac{2}{3}R$.

Uniform Wear Theory (UWT) I

- **Core Hypothesis**: Rate of surface wear is constant across the contact area.
- **Archard's Wear Relationship**: Wear rate $\propto p \cdot v$. Since sliding speed $v = \omega r$, uniform wear implies:

$$p \cdot r = C \quad \text{(constant)}$$

- Maximum pressure occurs at the inner radius where radius is smallest.
- **Axial Load Relation**:

$$W = \int_0^R p \cdot 2\pi r \, dr = 2\pi \int_0^R (p \cdot r) \, dr = 2\pi CR \implies C = \frac{W}{2\pi R}$$

- **Total Frictional Torque Derivation**:

$$T = \int_0^R 2\pi\mu(p \cdot r) r \, dr = 2\pi\mu C \int_0^R r \, dr = 2\pi\mu C \left[\frac{R^2}{2}\right] = \pi\mu C R^2$$

Uniform Wear Theory (UWT) II

- Substituting $C = \frac{W}{2\pi R}$ yields:

$$T = \frac{1}{2}\mu W R$$

- **Equivalent Mean Friction Radius**: $R_m = \frac{1}{2}R$.

Comparative Analysis: UPT vs. UWT I

- **Physical Transition**: New surfaces operate under UPT. Differential sliding speed ($v = \omega r$) causes faster wear at outer radii, shifting the pressure profile inward until UWT governs steady-state operation.

Table: Comparison of Uniform Pressure Theory (UPT) and Uniform Wear Theory (UWT)

Parameter / Criterion & **Uniform Pressure (UPT)** & **Uniform Wear (UWT)**
Surface Condition & New, unused surfaces & Worn-in, aged surfaces
Pressure Profile & $p =$ constant & $p \cdot r = C$ (variable p)
Peak Pressure & Constant across radius & Max at inner radius
Pivot Torque (T) & $\frac{2}{3}\mu WR$ & $\frac{1}{2}\mu WR$
Mean Friction Radius (R_m) & $\frac{2}{3}R \approx 0.67R$ & $\frac{1}{2}R = 0.50R$
Design Application & Conservative for Bearings & Conservative for Clutches

- **Engineering Design Rules**:

Comparative Analysis: UPT vs. UWT II

- **Clutches**: Designed under **UWT** because it yields lower torque capacity ($T_{\text{UWT}} < T_{\text{UPT}}$), preventing unexpected slippage under worn conditions.
- **Bearings**: Designed under **UPT** because it predicts higher frictional power loss, ensuring conservative thermal sizing.

Flat Collar Bearings Analysis I

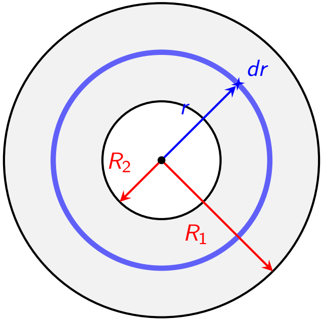
- **Geometry**: Annular contact ring defined by inner radius R_2 and outer radius R_1.
- Total contact area: $A = \pi(R_1^2 - R_2^2)$.
- **Frictional Torque under UPT**:
  - Pressure: $p = \frac{W}{\pi(R_1^2 - R_2^2)}$.
  - Torque integration from R_2 to R_1:

$$T = \int_{R_2}^{R_1} 2\pi\mu p r^2 \, dr = \frac{2}{3}\mu W \left(\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2} \right)$$

- **Frictional Torque under UWT**:
  - Load integration: $W = \int_{R_2}^{R_1} 2\pi C \, dr = 2\pi C(R_1 - R_2) \implies C = \frac{W}{2\pi(R_1 - R_2)}$.
  - Torque integration:

$$T = \int_{R_2}^{R_1} 2\pi\mu C r \, dr = \pi\mu C(R_1^2 - R_2^2) = \frac{1}{2}\mu W(R_1 + R_2)$$

Flat Collar Bearing Geometry I



Collar Area $A = \pi(R_1^2 - R_2^2)$

- Inner radius R_2 corresponds to shaft diameter; outer radius R_1 defines collar flange limit.
- Annular structure eliminates zero-velocity center singularity present in solid pivots.

Conical Pivots and Cone Clutches I

- **Wedge Effect Mechanics**:
  - Slanted contact surface inclined at semi-cone angle α relative to shaft axis.
  - Normal reaction on elemental surface: $dN_n = \frac{dN}{\sin\alpha} = \frac{p \cdot dA}{\sin\alpha}$.
  - Normal force increases by wedge factor $\frac{1}{\sin\alpha} = \csc\alpha$.
- **Conical Pivot Bearing Torque Formulae**:
  - Under UPT: $T = \frac{2}{3}\mu WR\csc\alpha$.
  - Under UWT: $T = \frac{1}{2}\mu WR\csc\alpha$.
- **Cone Clutch Torque Formulae (Annular limits R_2 to R_1)**:
  - Under UPT: $T = \frac{2}{3}\mu W\left(\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right)\csc\alpha$.
  - Under UWT: $T = \frac{1}{2}\mu W(R_1 + R_2)\csc\alpha$.

Frictional Power Loss & Thermal Design I

- **Power Loss Calculation**:
  - Shaft angular velocity: $\omega = \frac{2\pi N}{60}$ rad/s (N in RPM).
  - Power dissipated by friction:

$$P = T \cdot \omega = \frac{2\pi NT}{60} \quad \text{[Watts]}$$

- **Thermal Dissipation Considerations**:
  - Frictional power is directly converted into thermal energy ($Q_g = P$).
  - Excess heat induces lubricant degradation, thermal expansion, and potential surface seizure.
  - Operating design limit enforced by maximum permissible pressure $p_{\max}$ and allowable operating temperature.

Summary & Key Formulae Reference I

Table: Comprehensive Summary of Frictional Torque Formulae

Machine Element	**Uniform Pressure (UPT)**	**Uniform Wear (UWT)**
Flat Pivot (R)	$T = \frac{2}{3}\mu WR$	$T = \frac{1}{2}\mu WR$
Flat Collar (R_1, R_2)	$T = \frac{2}{3}\mu W\left(\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right)$	$T = \frac{1}{2}\mu W(R_1 + R_2)$
Conical Pivot (R, α)	$T = \frac{2}{3}\mu WR \csc\alpha$	$T = \frac{1}{2}\mu WR \csc\alpha$
Cone Clutch (R_1, R_2, α)	$T = \frac{2}{3}\mu W\left(\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right)\csc\alpha$	$T = \frac{1}{2}\mu W(R_1 + R_2)\csc\alpha$

Lecture 19: Overview and Objectives I

Course & Topic Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit:** Bearings, Clutches, Brake & Dynamometer
- **Lecture 19:** 3.1 Friction, Uniform Pressure and Wear Theories
- **Instructor:** Mechanical Engineering Professor

Lecture 19: Overview and Objectives II

Learning Objectives

1. Understand the principles of Coulomb dry friction in mechanical machine elements.
2. Analyze thrust bearings including flat pivot, conical pivot, and collar bearings.
3. Derive frictional torque formulas under **Uniform Pressure Theory (UPT)**.
4. Derive frictional torque formulas under **Uniform Wear Theory (UWT)** using Reye's Hypothesis.
5. Compare UPT vs. UWT for design applications in friction clutches and thrust bearings.

Fundamentals of Dry (Coulomb) Friction

Coulomb's Laws of Dry Friction

Frictional resistance between unlubricated contact surfaces of machine components is governed by Coulomb's friction laws:

- **Friction Force Formula:** The limiting frictional force F_f is directly proportional to the normal reaction force R_N:

$$F_f = \mu R_N$$

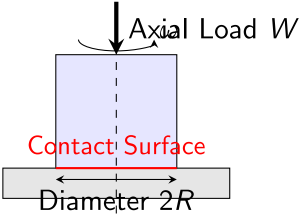
  where μ is the coefficient of friction.
- **Direction of Action:** Frictional force acts tangentially along the contact plane, opposing relative motion or the tendency of relative motion.
- **Static vs. Dynamic Friction:** Limiting static friction coefficient (μ_s) is slightly higher than kinetic/sliding friction coefficient (μ_k). Machine design calculations generally use the sliding coefficient μ.

Engineering Impact of Friction

- Frictional torque leads to mechanical power dissipation ($P = T\omega$), heat generation, and surface wear.
- In **clutches** and **brakes**, friction is intentionally harnessed to transmit torque or dissipate kinetic energy.
- In **bearings**, friction is minimized, but thrust loads require accurate torque calculation for efficiency and thermal analysis.

Classification of Thrust Bearings

- **Pivot Bearings:** Support axial thrust load W at the end of a rotating shaft. The contact surface can be flat or conical.
- **Collar Bearings:** Support axial shaft loads through one or more annular collars integral to or mounted onto the shaft.
- **Operational Requirement:** Transmit axial load W while shaft rotates at angular speed ω with minimum wear.



Flat Pivot Bearing Geometry

Pressure Distribution Assumption

Calculating total frictional torque T requires specific assumptions regarding pressure distribution across the contact interface: **Uniform Pressure** (new surfaces) or **Uniform Wear** (run-in surfaces).

Uniform Pressure Theory (UPT) I

Fundamental Assumption

UPT assumes constant intensity of normal pressure p across the entire contact area A:

$$p = \frac{W}{A} = \text{Constant}$$

Valid for **newly machined surfaces** and brand new bearings/clutches before initial wear occurs.

Uniform Pressure Theory (UPT) II

Derivation for Flat Pivot Bearing (Radius R)

- Pressure intensity: $p = \frac{W}{\pi R^2}$.
- Consider an elemental ring of radius r and width dr:

$$\text{Elemental Area } dA = 2\pi r\, dr$$

$$\text{Normal Force on Ring } dW = p \cdot dA = p \cdot (2\pi r\, dr)$$

- Frictional force on elemental ring: $dF_f = \mu\, dW = 2\pi\mu pr\, dr$.
- Frictional torque on elemental ring: $dT = dF_f \cdot r = 2\pi\mu pr^2\, dr$.
- Total Frictional Torque T:

$$T = \int_0^R 2\pi\mu pr^2\, dr = 2\pi\mu p\left[\frac{R^3}{3}\right] = 2\pi\mu\left(\frac{W}{\pi R^2}\right)\frac{R^3}{3} = \frac{2}{3}\mu WR$$

Uniform Wear Theory (UWT)

Reye's Hypothesis & Uniform Wear Condition

Wear rate is proportional to the product of pressure p and rubbing velocity v (Reye's Hypothesis):

$$\text{Wear Rate} \propto p \cdot v$$

Since rubbing velocity $v = r\omega$, for uniform wear across the contact surface:

$$p \cdot (r\omega) = \text{Constant} \implies p \cdot r = C \quad (\text{Constant})$$

Pressure is maximum at the inner radius ($r \to 0$) and minimum at the outer radius.

Derivation for Flat Pivot Bearing (Radius R)

- Total Axial Load W:

$$W = \int_0^R p \cdot 2\pi r\, dr = \int_0^R \frac{C}{r} \cdot 2\pi r\, dr = 2\pi C \int_0^R dr = 2\pi CR \implies C = \frac{W}{2\pi R}$$

- Elemental Frictional Torque dT:

$$dT = \mu \cdot dW \cdot r = \mu \left(\frac{C}{r} \cdot 2\pi r\, dr \right) r = 2\pi \mu C r\, dr$$

- Total Frictional Torque T:

$$T = \int_0^R 2\pi \mu C r\, dr = 2\pi \mu C \left[\frac{R^2}{2} \right] = 2\pi \mu \left(\frac{W}{2\pi R} \right) \frac{R^2}{2} = \frac{1}{2} \mu W R$$

Flat Pivot Bearings: UPT vs. UWT I

Comparison Summary for Flat Pivot Bearing (Radius R, Load W)

Parameter	**Uniform Pressure (UPT)**	**Uniform Wear (UWT)**
Pressure State	$p = \text{Constant} = \frac{W}{\pi R^2}$	$p \cdot r = C = \frac{W}{2\pi R}$
Frictional Torque (T)	$\frac{2}{3}\mu WR \approx 0.67\mu WR$	$\frac{1}{2}\mu WR = 0.50\mu WR$
Mean Radius (r_m)	$\frac{2}{3}R$	$\frac{1}{2}R$
Power Loss ($P = T\omega$)	Higher ($1.33\times$ UWT)	Lower ($0.75\times$ UPT)
Surface Condition	New / Un-worn surfaces	Run-in / Worn surfaces

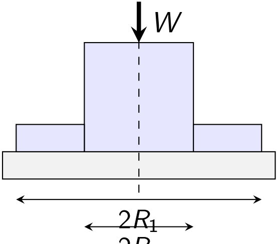
Flat Pivot Bearings: UPT vs. UWT II

Key Design Takeaways

- UPT predicts a 33% higher frictional torque and power loss than UWT for flat pivots.
- Bearings transition rapidly from UPT when new to UWT after initial operation due to localized wear at larger radii where sliding velocity v is highest.

Annular Contact Geometry

Collar bearings and flat plate clutches have an annular contact area bounded by inner radius R_2 and outer radius R_1.



Collar Bearing Geometry

Frictional Torque Formulas

- **Uniform Pressure Theory (UPT):**

$$T = \frac{2}{3}\mu W \left[\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right] \implies r_m = \frac{2}{3}\left[\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right]$$

- **Uniform Wear Theory (UWT):**

$$T = \frac{1}{2}\mu W(R_1 + R_2) \implies r_m = \frac{R_1 + R_2}{2}$$

- **Number of Contact Pairs (n):** For multi-plate clutches with n pairs of contact surfaces, $T_{\text{total}} = n \cdot T$.

Frictional Torque in Conical Pivot Bearings I

Conical Pivot Geometry

- A conical pivot features a shaft end shaped as a cone with semi-cone angle α resting in a conical seat.
- The normal pressure p_n acts perpendicular to the inclined conical surface.
- Relationship between axial load element dW and normal force element dR_N:

$$dW = dR_N \sin\alpha \implies dR_N = \frac{dW}{\sin\alpha} = dW \csc\alpha$$

Frictional Torque in Conical Pivot Bearings II

Torque Equations for Conical Pivot (Shaft Radius R)

- **Uniform Pressure Theory (UPT):**

$$T = \frac{2}{3}\mu W R \csc\alpha$$

- **Uniform Wear Theory (UWT):**

$$T = \frac{1}{2}\mu W R \csc\alpha$$

Analysis

Since $\csc\alpha > 1$ for $\alpha < 90^\circ$, conical pivots produce higher friction torque than flat pivots of equal radius R. When $\alpha = 90^\circ$, formulas reduce to flat pivot equations.

Multi-Collar Thrust Bearings

Purpose of Multi-Collar Construction

- Used when a single collar would require an excessively large outer diameter R_1 to support large axial thrust loads W.
- Splitting load across n collars reduces intensity of contact pressure p, preventing oil film breakdown and excessive local wear.

Load Distribution and Torque Calculation

- Let n be the number of identical collars.
- Axial load per collar: $W_c = \frac{W}{n}$.
- Pressure under UPT:

$$p = \frac{W}{n\pi(R_1^2 - R_2^2)}$$

- Total Frictional Torque for n collars under UPT:

$$T_{\text{total}} = n \times T_{\text{single collar}} = n\left[\frac{2}{3}\mu\left(\frac{W}{n}\right)\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right] = \frac{2}{3}\mu W\left[\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right]$$

- Total Frictional Torque under UWT:

$$T_{\text{total}} = \frac{1}{2}\mu W(R_1 + R_2)$$

Key Result

Total frictional torque is **independent of the number of collars** n for both UPT and UWT. Adding collars reduces surface pressure without increasing overall frictional torque!

Practical Design: Clutches vs. Bearings I

Design Conventions: UPT vs. UWT

- **Friction Clutch Design (Conservative Approach):**
  - Friction clutches are designed using **Uniform Wear Theory (UWT)**.
  - Reason: UWT yields a lower calculated torque capacity ($T_{\text{UWT}} < T_{\text{UPT}}$). Designing for UWT guarantees that the clutch will safely transmit required torque even after it is worn in.
- **Thrust Bearing Design:**
  - **Power Loss / Thermal Analysis:** Evaluated using **UWT** as bearings operate mostly in a run-in condition.
  - **Maximum Initial Pressure:** Evaluated using **UPT** when the bearing is brand new to prevent immediate surface crushing or seizure.

Practical Design: Clutches vs. Bearings II

Summary of Engineering Recommendations

1. Apply UPT for brand new, un-worn machinery components.
2. Apply UWT for run-in, operational machinery and conservative torque capacity calculations.
3. Ensure adequate lubrication to prevent dry friction wear regimes in continuous thrust bearings.

Lecture Overview & Learning Objectives I

Course Details

- **Course:** Theory of Machines and Mechanisms (GTU Code: DI03000141)
- **Unit:** Bearings, Clutches, Brakes & Dynamometers
- **Topic:** Sliding Contact Bearings and Friction-Induced Power Loss

Lecture Overview & Learning Objectives II

Learning Objectives

- Understand the fundamental classification of mechanical bearings (Sliding vs. Rolling, Radial vs. Thrust).
- Derive frictional torque equations for flat pivot bearings using Uniform Pressure Theory (UPT) and Uniform Wear Theory (UWT).
- Extend UPT and UWT principles to single and multi-collar thrust bearings.
- Analyze hydrodynamic lubrication in journal bearings using Petroff's equation.
- Evaluate power loss, heat generation (H_g), and heat dissipation (H_d) for thermal equilibrium design.

Classification of Bearings I

Primary Classification by Contact Nature

- **Sliding Contact Bearings:** High surface-to-surface contact where relative motion is pure sliding. Examples include journal bearings, pivot bearings, and collar bearings.
- **Rolling Contact Bearings (Anti-friction):** Utilize intermediate rolling elements (balls or rollers) to substitute sliding friction with rolling friction, significantly lowering starting torque.

Classification of Bearings II

Classification by Direction of Load Supported

- **Radial Bearings:** Support loads acting perpendicular to the axis of rotation of the shaft (e.g., standard journal bearing supporting a horizontal shaft).
- **Thrust Bearings:** Support axial thrust loads acting parallel to the rotational axis (e.g., pivot bearings, collar bearings).

Flat Pivot Bearing: Uniform Pressure Theory (UPT) I

Assumptions & Applicability

- Assumes uniform intensity of pressure p across the entire circular contact area ($p = \text{constant} = \frac{W}{\pi R^2}$).
- Valid primarily for **newly machined, perfectly flat, and rigid** bearing surfaces before any wear occurs.

Flat Pivot Bearing: Uniform Pressure Theory (UPT) II

Derivation of Frictional Torque

- Consider an elemental ring of radius r and radial thickness dr:

$$dA = 2\pi r\, dr, \quad dW = p \cdot dA = 2\pi pr\, dr$$

- Elemental frictional force and torque:

$$dF = \mu\, dW = 2\pi\mu pr\, dr \implies dT = dF \cdot r = 2\pi\mu pr^2\, dr$$

- Integrating over the full radius from 0 to R:

$$T = \int_0^R 2\pi\mu pr^2\, dr = 2\pi\mu p\left[\frac{R^3}{3}\right] = \frac{2}{3}\mu WR$$

- **Power Loss:** $P = T \cdot \omega = T\left(\frac{2\pi N}{60}\right)$

Flat Pivot Bearing: Uniform Wear Theory (UWT) I

Assumptions & Applicability

- Wear rate is proportional to the product of pressure p and sliding velocity v ($p \cdot v =$ constant).
- Since linear velocity $v = \omega r$, uniform wear implies:

$$p \cdot r = C \quad \text{(constant)}$$

- Applicable to **worn-in or run-in** bearings. Maximum pressure occurs at $r \to 0$ (p_{max}) and minimum at $r = R$ (p_{min}).

Flat Pivot Bearing: Uniform Wear Theory (UWT) II

Derivation of Frictional Torque

- Total load carried:

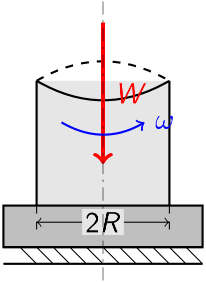
$$W = \int_0^R p(2\pi r\,dr) = 2\pi C \int_0^R dr = 2\pi CR \implies C = \frac{W}{2\pi R}$$

- Integrating elemental torque $dT = 2\pi\mu Cr\,dr$:

$$T = \int_0^R 2\pi\mu Cr\,dr = 2\pi\mu C\left[\frac{R^2}{2}\right] = \frac{1}{2}\mu WR$$

- **Comparison:** UWT gives $T = 0.5\mu WR$, which is 25% lower than UPT ($0.67\mu WR$). UWT is universally used for safe load-capacity design of aged bearings.

Schematic and Analysis of Flat Pivot Bearing I



Key Features of Flat Pivot Bearing

- Vertical axial load W acts along the shaft centerline.
- Shaft rotates at angular velocity ω against stationary step surface.
- Contact area is a solid disk of radius R.

Flat Collar Bearing Mechanics I

Annular Contact Geometry

- Used when a shaft passes continuously through a bearing support.
- Defined by inner shaft radius r_i and outer collar radius r_o.
- Net bearing area: $A = \pi(r_o^2 - r_i^2)$.

Flat Collar Bearing Mechanics II

Frictional Torque Equations

- **Uniform Pressure Theory (UPT):**

$$p = \frac{W}{\pi(r_o^2 - r_i^2)}, \quad T = \frac{2}{3}\mu W \left[\frac{r_o^3 - r_i^3}{r_o^2 - r_i^2}\right]$$

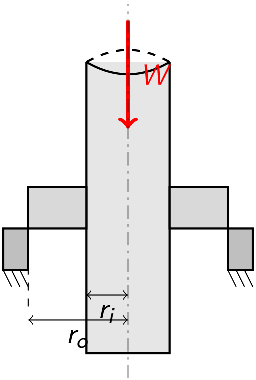
- **Uniform Wear Theory (UWT):**

$$p \cdot r = C = \frac{W}{2\pi(r_o - r_i)}, \quad T = \frac{1}{2}\mu W(r_o + r_i) = \mu W r_m$$

  where $r_m = \frac{r_o + r_i}{2}$ is the mean radius.

- **Multi-Collar Bearings:** If n identical collars are used, total axial load is split equally, while total torque T remains governed by total axial load W.

Schematic and Geometry of Flat Collar Bearing I



Schematic and Geometry of Flat Collar Bearing II

Geometrical Characteristics

- Inner radius r_i corresponds to the shaft radius.
- Outer radius r_o bounds the annular collar extension.
- Annular bearing area withstands high axial loads while allowing shaft continuation.

Journal Bearings: Viscous Shear and Petroff's Equation

Hydrodynamic Lubrication Overview

- Journal of radius r rotates inside a stationary bushing of radius R_b with radial clearance $c = R_b - r$.
- Fluid film of dynamic viscosity μ completely separates metal surfaces under full-film hydrodynamic conditions.

Petroff's Equation Derivation

- Assuming concentric journal alignment (zero eccentricity ratio), velocity gradient across film clearance c:

$$\frac{dv}{dy} = \frac{v}{c} = \frac{2\pi rn}{c} \quad (n = \text{rotational speed in rps})$$

- Viscous shear stress (τ) and total shear force (F):

$$\tau = \mu\frac{dv}{dy} = \frac{2\pi\mu rn}{c}, \quad F = \tau \cdot (2\pi rL) = \frac{4\pi^2\mu r^2 nL}{c}$$

- Frictional torque $T = F \cdot r$ and coefficient of friction f:

$$T = \frac{4\pi^2\mu nLr^3}{c}, \quad f = 2\pi^2\left(\frac{\mu n}{p}\right)\left(\frac{r}{c}\right)$$

where projected bearing pressure $p = \frac{W}{2rL}$.

Friction-Induced Power Loss and Heat Generation I

Frictional Power Loss

- Mechanical work expended per second in overcoming bearing friction is converted into heat:

$$P = T \cdot \omega = T\left(\frac{2\pi N}{60}\right) \quad \text{[Watts]}$$

- Alternatively, using coefficient of friction f and linear surface velocity $v = \frac{\pi d N}{60}$:

$$H_g = P = f \cdot W \cdot v \quad \text{[J/s or W]}$$

Friction-Induced Power Loss and Heat Generation II

Heat Generation Dynamics (H_g)

- Higher operating speeds (N) and heavy loads (W) directly increase heat generation rate H_g.
- Viscosity μ decreases exponentially with rising oil temperature, potentially leading to film breakdown if uncontrolled.

Thermal Equilibrium and Heat Dissipation I

Heat Dissipation Capacity (H_d)

- Heat dissipated by bearing housing to surroundings via convection and radiation:

$$H_d = C \cdot A \cdot (T_b - T_a) \quad \text{[Watts]}$$

where:

- C = Heat dissipation coefficient ($\mathrm{W/(m^2 \cdot {}^\circ C)}$)
- $A = L \cdot d$ = Projected area of the bearing ($\mathrm{m^2}$)
- T_b = Operating temperature of bearing surface ($^\circ$C)
- T_a = Ambient air temperature ($^\circ$C)

Thermal Equilibrium and Heat Dissipation II

Thermal Equilibrium Criterion

- Under steady-state operation:

$$H_g = H_d$$

- If $H_g > H_d$, oil temperature rises continuously, causing lubricant breakdown and risk of bearing seizure. Forced oil-cooling with external heat exchangers becomes mandatory.

Comparison Summary and Engineering Design Guidelines I

Summary Comparison: Pivot and Collar Bearings

Bearing Type	**Uniform Pressure Theory (UPT)**	**Uniform Wear Theory (UWT)**
Flat Pivot	$T = \frac{2}{3}\mu WR$	$T = \frac{1}{2}\mu WR$
Flat Collar	$T = \frac{2}{3}\mu W \left[\frac{r_o^3 - r_i^3}{r_o^2 - r_i^2}\right]$	$T = \frac{1}{2}\mu W(r_o + r_i)$
Physical State	New / Unworn Bearings	Worn-in / Aged Bearings
Design Usage	Conservative Power Loss Sizing	Safe Load Capacity Design

Comparison Summary and Engineering Design Guidelines II

Key Engineering Takeaways

- **Theory Selection:** Always design load-carrying capacity based on UWT as it reflects realistic worn-in operating states.
- **Hydrodynamic Safety:** Ensure the Sommerfeld number $S = \left(\frac{r}{c}\right)^2 \left(\frac{\mu n}{p}\right)$ operates in the stable fluid-film lubrication regime.
- **Thermal Management:** Verify $H_d \geq H_g$ to maintain oil viscosity and prevent catastrophic surface contact.

Lecture Overview: 3.2 Bearings and Power Loss I

Course & Unit Details

- **Course**: Theory of Machines and Mechanisms (GTU Code: DI03000141)
- **Unit**: Bearings, Clutches, Brake & Dynamometer
- **Topic**: 3.2 Bearings and Power Loss – Classification, Friction Theories, and Power Loss Calculations

Lecture Overview: 3.2 Bearings and Power Loss II

Learning Objectives

- Classify lubrication regimes: Boundary, Mixed, Hydrodynamic, and Hydrostatic lubrication.
- Analyze hydrodynamic pressure generation and fluid wedge formation in journal bearings.
- Derive Petroff's equation for friction coefficient in lightly loaded journal bearings.
- Interpret McKee's experimental investigation and the Stribeck friction curve.
- Compute friction torque, power loss, and thermal dissipation in journal bearings.
- Evaluate power loss in flat pivot and collar thrust bearings using Uniform Pressure Theory (UPT) and Uniform Wear Theory (UWT).

Introduction to Bearings & Lubrication Regimes

Definition and Primary Function

A **bearing** is a stationary or moving machine element that supports and guides another moving element (journal or shaft), constraining relative motion to the desired degree while minimizing friction and surface wear.

Classification of Lubrication Regimes

- **Boundary Lubrication**: Occurs when lubricant film thickness (h) is smaller than surface roughness peak height (R_a). Direct solid-to-solid contact takes place at surface asperities. Friction coefficient is high ($f \approx 0.08 - 0.15$).
- **Mixed (Quasi-Hydrodynamic) Lubrication**: Intermediate regime where radial load is partially supported by fluid pressure and partially by direct contact of high asperities.
- **Hydrodynamic Lubrication**: Established when a continuous, viscous fluid film completely separates journal and bearing surfaces ($h_0 \gg R_a$). Friction is governed entirely by internal fluid shear ($f \approx 0.001 - 0.005$).
- **Hydrostatic Lubrication**: External high-pressure pump supplies lubricant at high pressure to separate surfaces before rotation begins, ensuring zero wear during start-up or zero-speed conditions.

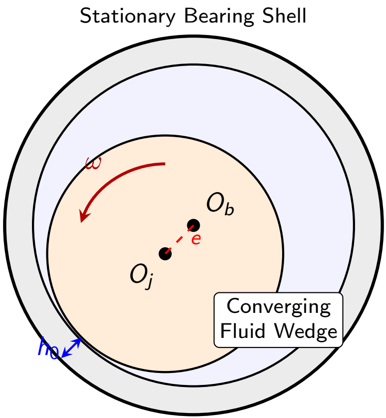
Hydrodynamic Journal Bearing Lubrication I

Hydrodynamic Wedging Principle

Relative sliding motion between non-parallel surfaces draws viscous fluid into a converging channel (fluid wedge), generating a hydrodynamic pressure field capable of supporting external radial load W.

Geometric Parameters

- O_b: Center of stationary bearing bore; O_j: Center of rotating journal.
- **Eccentricity** (e): Radial offset between O_b and O_j under load.
- **Radial Clearance** ($c = R_b - R_j$): Radial difference between bearing bore and journal.
- **Minimum Film Thickness** ($h_0 = c - e = c(1 - \epsilon)$), where $\epsilon = e/c$ is the eccentricity ratio ($0 \leq \epsilon < 1$).



Stationary Bearing Shell
ω
O_b
e
O_j
Converging Fluid Wedge
h_0

Friction in Sliding Contact Bearings I

Viscous Shearing Mechanism

Under full hydrodynamic lubrication, surface friction is eliminated. Frictional resistance to shaft rotation arises strictly from viscous shear stress within the lubricant film.

Petroff's Method Assumptions

- Concentric journal alignment ($e \approx 0$, light radial load condition).
- Uniform radial clearance c along the entire circumferential perimeter.
- Linear velocity gradient across the thin lubricant film thickness ($\frac{dv}{dy} = \frac{v}{c}$).
- Negligible end leakage of lubricant ($L/d \to \infty$).

Friction in Sliding Contact Bearings II

Shear Stress and Force Equations

- Surface velocity: $v = \omega r = \frac{2\pi rN}{60}$
- Newtonian shear stress: $\tau = \eta \cdot \frac{v}{c} = \eta \cdot \frac{2\pi rN}{60c}$
- Frictional shear force: $F_f = \tau \cdot A = \eta \left(\frac{2\pi rN}{60c}\right)(2\pi rL) = \frac{4\pi^2 \eta r^2 LN}{60c}$
- Frictional power loss is converted directly into heat, elevating oil operating temperature and reducing dynamic viscosity η.

Petroff's Equation for Coefficient of Friction

Derivation of Petroff's Formula

Expressing friction force in terms of coefficient of friction f and projected bearing load W:

$$F_f = f \cdot W$$

Projected bearing area $A_p = 2rL$, giving projected bearing pressure $p = \frac{W}{2rL} \implies W = 2prL$.
Equating expressions for F_f:

$$f(2prL) = \frac{4\pi^2 \eta r^2 LN}{60c}$$

Substituting rotational speed in rev/s ($n = N/60$):

$$f = 2\pi^2 \left(\frac{\eta n}{p}\right)\left(\frac{r}{c}\right)$$

Significance of Dimensionless Groups

- **Bearing Parameter** $\left(\frac{\eta n}{p}\right)$: Dimensionless operational variable combining viscosity η, speed n, and projected pressure p.
- **Clearance Ratio** $\left(\frac{r}{c}\right)$: Geometric scale factor, typically ranging from 500 to 1000 for industrial machinery.
- **Limitation**: Petroff's linear relation strictly applies to lightly loaded, high-speed bearings ($e \approx 0$). Heavily loaded bearings form asymmetric pressure distributions.

McKee's Empirical Investigation of Friction I

Experimental Mapping by McKee

Real journal bearings operate under eccentricity and fluid end leakage. McKee mapped friction coefficient f against operational bearing parameter $\frac{Zn}{p}$ across all lubrication regimes.

McKee's Empirical Investigation of Friction II

McKee's Empirical Formula

$$f = K_a \left(\frac{Zn}{p}\right)\left(\frac{r}{c}\right) + \Delta f$$

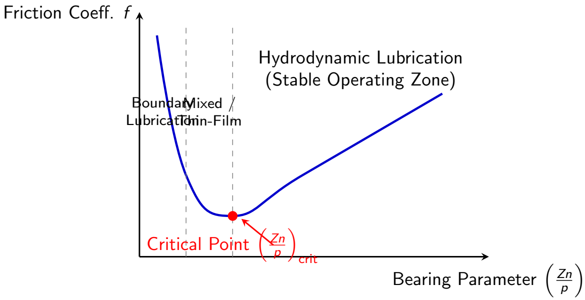
- Z: Dynamic viscosity in centipoise (cP).
- n: Rotational speed in revolutions per second (rps).
- p: Projected bearing pressure (N/mm^2 or psi).
- K_a: End-leakage structural constant ($K_a = 0.326 \times 10^{-2}$ for imperial units or 4.73×10^{-8} for SI units).
- Δf: Empirical correction factor (≈ 0.002) for boundary and end friction.

McKee's Empirical Investigation of Friction III

Critical Bearing Parameter

The minimum point of McKee's curve defines the transition from unstable thin-film lubrication to stable hydrodynamic lubrication. Designers select operating $\frac{Zn}{p}$ about 5 to 6 times this critical value.

McKee's Friction Curve (Stribeck Curve)



Lubrication Zone Characteristics

- **Boundary Zone**: Asperities contact directly; high f and rapid wear.
- **Mixed Zone**: Film develops; f drops steeply to a minimum ($\frac{df}{d(Zn/p)} < 0$, thermally unstable region).
- **Hydrodynamic Zone**: Full film separation; f increases linearly with speed ($\frac{df}{d(Zn/p)} > 0$, self-regulating stable region).

Power Loss & Thermal Equilibrium in Journal Bearings I

Friction Torque and Heat Generation

Friction Torque:

$$T_f = f \cdot W \cdot r$$

Heat Generated by Viscous Friction (H_g):

$$H_g = P_{\text{loss}} = T_f \cdot \omega = \frac{2\pi N T_f}{60} \quad \text{[Watts]}$$

Power Loss & Thermal Equilibrium in Journal Bearings II

Heat Dissipation Capacity

Heat Dissipated by Bearing Housing (H_d):

$$H_d = C \cdot A_d \cdot (t_b - t_a)$$

- C: Heat dissipation coefficient (8 to $20\,\mathrm{W/(m^2 \cdot {}^\circ C)}$ depending on air velocity).
- A_d: Housing dissipation surface area ($\approx 20 \times L \cdot d$).
- t_b: Bearing housing surface operating temperature ($^\circ$C).
- t_a: Ambient air temperature ($^\circ$C).

Power Loss & Thermal Equilibrium in Journal Bearings III

Thermal Equilibrium Condition

At steady state, $H_g = H_d$. If $H_g > H_d$, oil temperature rises, reducing viscosity η, which can break down the hydrodynamic film. External artificial cooling (oil circulating pumps with heat exchangers) is required when $H_g > H_d$.

Power Loss in Thrust Bearings (Pivot and Collar) I

Function of Axial Thrust Bearings

Thrust bearings support axial loads (W) along rotating shafts (e.g., marine propeller shafts, vertical turbines). Frictional torque is analyzed under two distinct tribological theories:

Parameter / Theory	**Uniform Pressure (UPT)**	**Uniform Wear (UWT)**
Physical Basis	New, un-worn bearing surface ($p = \text{const}$)	Worn-in running bearing ($p \cdot r = \text{const}$)
Contact Pressure	$p = \frac{W}{\pi(R_o^2 - R_i^2)}$	$p(r) = \frac{C}{r} = \frac{W}{2\pi(R_o - R_i)r}$
Collar Friction Torque (T_f)	$T_f = \frac{2}{3} fW \left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)$	$T_f = \frac{1}{2} fW(R_o + R_i)$
Flat Pivot Torque ($R_i = 0$)	$T_f = \frac{2}{3} fWR$	$T_f = \frac{1}{2} fWR$

Power Loss in Thrust Bearings (Pivot and Collar) II

Engineering Design Significance

Uniform Wear Theory predicts lower friction torque ($T_{f,\text{UWT}} \leq T_{f,\text{UPT}}$). Because running wear evens out surface contact pressure over time, UWT is universally adopted for designing worn-in thrust bearings.

Bearing Design Optimization & Parameter Selection I

Impact of Design Parameters on Journal Bearing Performance

Parameter Change	Load Capacity	Power Loss	Operating Temp.
Increase Viscosity ($\eta \uparrow$)	Increases	Increases	Increases
Increase Speed ($N \uparrow$)	Increases	Increases	Increases
Decrease Clearance ($c \downarrow$)	Increases	Increases	Increases
Increase Aspect Ratio ($L/d \uparrow$)	Increases	Increases	Moderate Increase

Bearing Design Optimization & Parameter Selection II

Optimization Rules

- Maximum oil film temperature must be kept below 80°C to prevent thermal degradation of additives.
- Recommended clearance ratio range: $500 \leq r/c \leq 1000$ ($c/r \approx 0.001$).
- Multi-collar thrust bearings distribute heavy axial loads across n collars, reducing average pressure and mitigating surface distortion.

Summary of Key Formulas & Design Guidelines I

Core Mathematical Summary

- Petroff's Formula: $f = 2\pi^2 \left(\frac{\eta n}{p}\right)\left(\frac{r}{c}\right)$
- McKee's Formula: $f = K_a \left(\frac{Zn}{p}\right)\left(\frac{r}{c}\right) + \Delta f$
- Frictional Power Loss: $P_{\text{loss}} = T_f \cdot \omega = \frac{2\pi N T_f}{60}$
- Heat Dissipation: $H_d = C \cdot A_d \cdot (t_b - t_a)$
- Thrust Collar Torque (UWT): $T_f = \frac{1}{2} f W (R_o + R_i)$
- Thrust Collar Torque (UPT): $T_f = \frac{2}{3} f W \left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)$

Summary of Key Formulas & Design Guidelines II

Key Design Takeaways

1. Maintain operating point well to the right of McKee's critical minimum ($5\text{–}6 \times (Zn/p)_{\text{crit}}$) to prevent thermal runaway.
2. Verify thermal equilibrium ($H_g = H_d$); provide external oil cooling if $H_g > H_d$.
3. Apply Uniform Wear Theory for conservative safety evaluation of run-in thrust collars and pivots.

Lecture 22: 3.3 Clutches I

Course Details

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit**: Bearings, Clutches, Brake & Dynamometer
- **Topic**: 3.3 Clutches
- **Target Audience**: Mechanical Engineering Students (Semester 3)

Lecture 22: 3.3 Clutches II

Key Learning Objectives

- Understand the mechanical function and classification of clutches.
- Differentiate between Uniform Pressure Theory (UPT) and Uniform Wear Theory (UWT).
- Derive frictional torque equations for single-plate, multi-plate, cone, and centrifugal clutches.
- Analyze elemental face geometry and force equilibrium in contact friction elements.
- Evaluate thermal dissipation and material selection criteria for friction linings.

Introduction to Clutches I

Definition

A **clutch** is a mechanical device used to connect or disconnect the driving shaft from the driven shaft at the will of the operator, without stopping the prime mover (engine or motor).

Introduction to Clutches II

Primary Functions & Features

- **Idle Operation**: Allows the vehicle or machine to come to a complete stop while the prime mover continues to run.
- **Gradual Engagement**: Ensures smooth, shock-free engagement of power transmission elements during acceleration.
- **Overload Protection**: Serves as a safety slip device when transmitted torque exceeds safe mechanical limits.
- **Thermal Capacity**: Requires high heat dissipation capabilities to handle energy converted due to frictional slip during engagement.

Categories of Clutches

Clutches are broadly categorized based on their torque transmission mechanism:

Clutch Category	Transmission Mode	Slip Capability	Key Applications
Positive Contact	Jaw / Claw Interlocking	Zero slip	Machine tools, Punch presses
Friction	Surface Friction	Gradual slip	Automobiles, Motorcycles
Hydraulic	Fluid Hydrodynamic	Inherent slip	Automatic transmissions
Electromagnetic	Magnetic Field Control	Controlled slip	Automation, Robotics

Table: Overview of Clutch Classifications

Classification of Clutches II

Key Characteristics

- **Positive Clutches**: Direct interlocking; cannot be engaged at high relative speeds.
- **Friction Clutches**: Surface contact; enables smooth start-up from rest under load.

Friction Clutches: Design Hypotheses I

Design Theories

The analysis of friction clutches relies on two fundamental hypotheses regarding normal pressure distribution across contacting surfaces:

1. Uniform Pressure Theory (UPT)

- Assumes normal pressure p is uniformly distributed over the entire contact area ($p =$ constant).
- Applies to **new, unused** friction plates with perfectly flat surfaces.

Friction Clutches: Design Hypotheses II

2. Uniform Wear Theory (UWT)

- Assumes rate of wear is uniform over the entire contact surface.
- Wear rate $\propto p \cdot v \propto p \cdot r \implies p \cdot r = C$ (constant).
- Applies to **worn or run-in** clutches after initial operating service.

Design Recommendation

Torque capacity predicted by UWT is lower than UPT ($T_{\text{UWT}} \leq T_{\text{UPT}}$). Therefore, **UWT is always preferred for conservative mechanical design**.

Geometry & Notation

Let r_1 = Outer radius of friction plate, r_2 = Inner radius of friction plate, W = Total axial thrust force, μ = Coefficient of friction, n = Number of active friction surfaces ($n = 2$ for standard single plate clutch).

Uniform Pressure Theory (UPT)

- Normal pressure: $p = \frac{W}{\pi(r_1^2 - r_2^2)}$
- Frictional Torque per surface: $T = \frac{2}{3}\mu W \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$

Single Plate Clutch: Mathematical Analysis II

Uniform Wear Theory (UWT)

- Pressure distribution constant: $C = p \cdot r = \frac{W}{2\pi(r_1 - r_2)}$
- Mean radius: $R_m = \frac{r_1 + r_2}{2}$
- Frictional Torque per surface: $T = \mu W R_m = \mu W \left(\frac{r_1 + r_2}{2}\right)$

Total Torque Capacity

$$T_{\text{total}} = n \cdot T$$

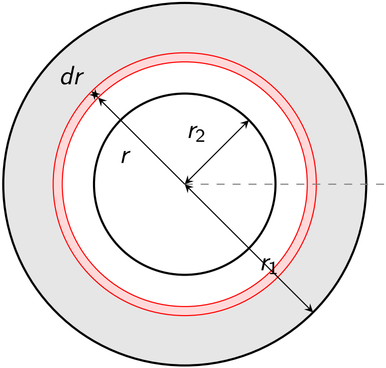
Friction Face & Elemental Ring Geometry I

Elemental Ring Analysis

Consider an elemental ring at radius r with width dr on the annular friction surface:

- Area of elemental ring: $dA = 2\pi r \, dr$
- Normal force on ring: $dW = p \cdot dA = 2\pi pr \, dr$
- Frictional force on ring: $dF = \mu \cdot dW = 2\pi \mu pr \, dr$
- Frictional torque on ring: $dT = dF \cdot r = 2\pi \mu pr^2 \, dr$

Friction Face & Elemental Ring Geometry II



Friction Lining and Elemental Ring Geometry

Multi-plate Clutch Systems I

Purpose & Construction

- Used when high torque must be transmitted within constrained radial dimensions (e.g., motorcycles, sports cars, automatic transmissions).
- Consists of alternating driving plates splined to the input hub and driven plates splined to the output housing.

Active Contact Surfaces

Let n_1 = Number of driving plates, n_2 = Number of driven plates.

$$n = n_1 + n_2 - 1$$

where n is the total number of active contacting friction surfaces.

Torque Transmission Capacity

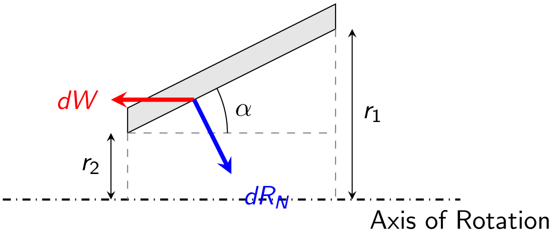
Assuming Uniform Wear Theory (UWT):

$$T_{\text{total}} = n \cdot \mu \cdot W \cdot R_m = n \cdot \mu \cdot W \cdot \left(\frac{r_1 + r_2}{2}\right)$$

Cone Clutch Geometry and Mechanics I

Working Principle

- Driver consists of a cone cup and driven element is a solid cone mating into the cup.
- **Semi-cone angle (α)**: Typically designed between 12.5° and 15°.
- If $\alpha < 12°$, the clutch risks **self-locking**, making disengagement difficult.



Cone Clutch Geometry and Mechanics II

Torque Capacity Equation (UWT)

$$T = \frac{\mu \cdot W \cdot R_m}{\sin\alpha} = \frac{\mu \cdot W}{\sin\alpha}\left(\frac{r_1 + r_2}{2}\right)$$

Note: Normal reaction $R_N = \frac{W}{\sin\alpha}$, significantly amplifying friction capacity.

Centrifugal Clutch Analysis I

Operating Principle

Engages automatically as engine speed increases, driven by centrifugal force acting on internal radial shoes.

Force Equilibrium per Shoe

- Centrifugal force at speed ω: $F_c = m \cdot \omega^2 \cdot r_g$
- Inward spring force at engagement speed ω_0: $F_s = m \cdot \omega_0^2 \cdot r_g$
- Net radial force on drum rim: $P = F_c - F_s = m \cdot r_g(\omega^2 - \omega_0^2)$

Torque Transmission Capacity

For n centrifugal shoes inside a drum of internal radius R_{drum}:

$$T_{\text{total}} = n \cdot \mu \cdot (F_c - F_s) \cdot R_{\text{drum}} = n \cdot \mu \cdot m \cdot r_g(\omega^2 - \omega_0^2) \cdot R_{\text{drum}}$$

Clutch Materials & Thermal Considerations I

Friction Material Requirements

- High and stable coefficient of friction ($\mu \approx 0.35 - 0.50$).
- High thermal conductivity and temperature resistance to prevent friction fade.
- Excellent wear resistance and mechanical strength under high shear forces.

Friction Lining Types

- **Organic Composites**: Standard automotive clutches ($\mu \approx 0.35$).
- **Sintered Metallic / Ceramic**: Heavy-duty industrial and racing applications ($\mu \approx 0.45$).
- **Carbon-Carbon Composites**: High-performance aerospace applications.

Clutch Materials & Thermal Considerations II

Thermal Dissipation & Slip Energy

Energy dissipated as heat during slip engagement:

$$Q = \int_0^{t_{\text{slip}}} T(\Delta\omega)\, dt \approx \frac{1}{2} I (\Delta\omega)^2$$

Wet clutches operating in oil baths enhance cooling rates and wear life significantly.

Summary & Comparison of Clutch Types I

Comparative Summary Table

Clutch Type	Key Feature	Torque Expression (UWT)	Primary Advantage
Single Plate	Simple design	$T = 2\mu W R_m$	Excellent heat dissipation
Multi-plate	Multiple surfaces	$T = n\mu W R_m$	Compact radial dimensions
Cone Clutch	Wedging action	$T = \frac{\mu W R_m}{\sin\alpha}$	High force multiplication
Centrifugal	Automatic engagement	$T = n\mu(F_c - F_s)R_{\text{drum}}$	Automatic speed control

Table: Comprehensive Comparison of Friction Clutch Mechanics

Final Design Takeaways

- Always utilize Uniform Wear Theory (UWT) for safe conservative engineering design.
- Select multi-plate designs when radial envelope is restricted.
- Ensure semi-cone angle $\alpha > 12^\circ$ in cone clutches to avoid mechanical self-locking.

Course Details

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit**: Bearings, Clutches, Brakes & Dynamometers
- **Lecture 23**: 3.3 Clutches

Course Overview & Lecture Details II

Topics Covered

- Working principles and functional requirements of clutches
- Detailed classification of mechanical clutches
- Construction and mechanics of single-plate and multi-plate clutches
- Design equations based on Uniform Pressure Theory (UPT) and Uniform Wear Theory (UWT)
- Mechanics of Cone Clutches and Centrifugal Clutches
- Selection criteria for friction materials

Introduction & Working Principle I

Definition of a Clutch

A clutch is a mechanical device used to connect or disconnect a driving shaft from a driven shaft, allowing the driven shaft to be started or stopped at will without stopping the prime mover (driving shaft).

Introduction & Working Principle II

Core Functional Requirements

- **Torque Transmission**: Must transmit maximum engine torque without slipping under normal operating conditions.
- **Gradual Engagement**: Must provide smooth engagement without shock or vibration to transmission components.
- **Heat Dissipation**: Must effectively dissipate thermal energy generated due to friction during relative slipping.
- **Dynamic Balance**: Rotational components must be statically and dynamically balanced for high-speed operation.
- **Low Inertia**: Driven member should have low rotational inertia to facilitate quick gear shifting.

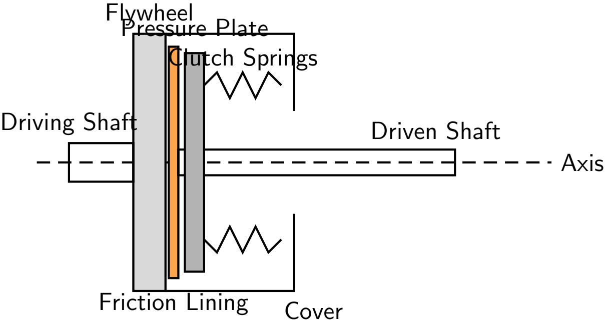
Classification of Clutches I

Category	Mechanism / Types	Key Characteristics
Positive Contact	Jaw clutches (Square, Spiral)	No slip; engaged at zero relative speed
Friction Clutches	Single-plate, Multi-plate, Cone	Allows engagement at relative speeds
Hydraulic / Fluid	Fluid coupling, Torque converter	Hydrodynamic power transfer; smooth drive
Electromagnetic	Magnetic friction, Particle chains	Actuated electrically via magnetic field

Friction Clutch Variants

- **Dry Clutches**: High coefficient of friction ($\mu \approx 0.3 - 0.45$), air-cooled, widely used in automobiles.
- **Wet Clutches**: Oil-immersed, lower coefficient of friction ($\mu \approx 0.05 - 0.1$), superior heat dissipation and longer wear life.

Single Plate Clutch Architecture I



Single Plate Clutch Architecture II

Component Functions

- **Flywheel**: Mounts to the engine crankshaft; acts as the primary driving friction surface.
- **Clutch Plate / Friction Lining**: Splined to the driven shaft; equipped with friction facings on both sides.
- **Pressure Plate**: Clamps the clutch plate against the flywheel using axial helical or diaphragm springs.

Friction Clutches: Core Design Assumptions I

Boundary Conditions in Clutch Analysis

When analyzing torque transmission capacity, two classical theories are applied based on the physical state of the friction surfaces:

1. Uniform Pressure Theory (UPT)

- Assumes the normal pressure distribution p is uniform over the entire contact area: $p = \text{constant}$.
- Applicable to **new, perfectly flat, and rigid** clutch plates.

Friction Clutches: Core Design Assumptions II

2. Uniform Wear Theory (UWT)

- Based on the physical law that wear rate is proportional to frictional work done:

$$\text{Rate of Wear} \propto p \cdot v = \text{constant}$$

- Since linear velocity $v = \omega \cdot r$, the governing relationship becomes:

$$p \cdot r = C \quad \text{(constant)}$$

- Applicable to **worn or in-service** clutches where initial running-in has occurred.

Uniform Pressure Theory (UPT) Equations I

Derivation Parameters

Let r_1 = outer radius of friction plate, r_2 = inner radius of friction plate, p = uniform normal pressure, μ = coefficient of friction, W = total axial thrust force.

Axial Thrust Force (W)

Integrating elemental normal force over the annular ring:

$$W = \int_{r_2}^{r_1} 2\pi p\, r\, dr = \pi p(r_1^2 - r_2^2)$$

Uniform Pressure Theory (UPT) Equations II

Frictional Torque Capacity (T)

Integrating elemental friction torque $dT = \mu(2\pi p\, r\, dr)r$:

$$T = \int_{r_2}^{r_1} 2\pi\mu p\, r^2\, dr = \frac{2}{3}\pi\mu p(r_1^3 - r_2^3)$$

Substituting $p = \frac{W}{\pi(r_1^2 - r_2^2)}$ yields:

$$T = \mu W R_m \quad \text{where} \quad R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$

Uniform Wear Theory (UWT) Equations I

Pressure Distribution

Under $p \cdot r = C$, maximum pressure $p_{\max}$ occurs at the inner radius r_2:

$$C = p_{\max} \cdot r_2$$

Axial Thrust Force (W)

$$W = \int_{r_2}^{r_1} 2\pi p\, r\, dr = 2\pi C \int_{r_2}^{r_1} dr = 2\pi C(r_1 - r_2) = 2\pi p_{\max} r_2 (r_1 - r_2)$$

Uniform Wear Theory (UWT) Equations II

Frictional Torque Capacity (T)

$$T = \int_{r_2}^{r_1} 2\pi\mu C r \, dr = \pi\mu C(r_1^2 - r_2^2)$$

Substituting $C = \frac{W}{2\pi(r_1 - r_2)}$ yields:

$$T = \mu W R_m \quad \text{where} \quad R_m = \frac{r_1 + r_2}{2}$$

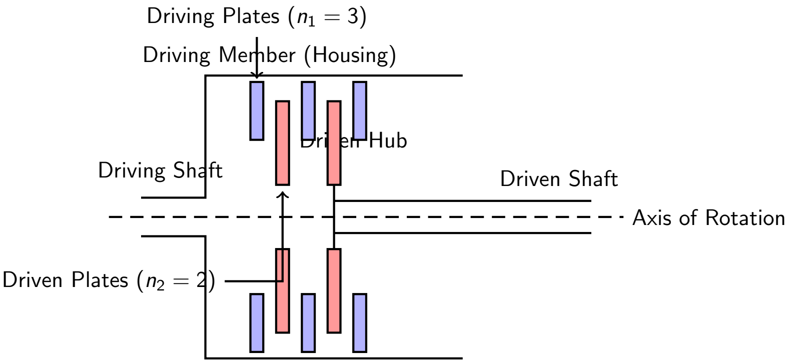
Comparison Summary: UPT vs. UWT I

Parameter	Uniform Pressure Theory (UPT)	Uniform Wear Theory (UWT)
Pressure Assumption	$p =$ constant across radius	$p \cdot r = C$ ($p_{\max}$ at inner radius r_2)
Plate Condition	Brand new, rigid, unworn plates	In-service, worn, flexible plates
Mean Radius (R_m)	$R_m = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$	$R_m = \frac{r_1 + r_2}{2}$
Torque Capacity	Predicts higher torque capacity	Predicts lower torque capacity
Design Safety	Less conservative	More conservative (Preferred for design)

Design Rule

Engineers must always base clutch design and sizing on **Uniform Wear Theory (UWT)** because it predicts a lower, safer torque capacity over the operational lifespan of the friction surfaces.

Multi-Plate Friction Clutch I



Multi-Plate Friction Clutch II

Active Contact Surfaces

Let n_1 = number of driving plates, n_2 = number of driven plates.

$$n = n_1 + n_2 - 1$$

Total Frictional Torque Transmitted:

$$T = n \cdot \mu \cdot W \cdot R_m$$

Cone Clutch Mechanics I

Working Geometry

A cone clutch utilizes internal and external conical surfaces with semi-cone angle α.

Force Equilibrium

Resolving axial force W against normal force R_N and friction during engagement:

$$W = R_N \sin\alpha + \mu R_N \cos\alpha = R_N(\sin\alpha + \mu\cos\alpha)$$

When disengaging or under steady engagement (neglecting disengagement friction force):

$$W \approx R_N \sin\alpha \implies R_N = \frac{W}{\sin\alpha}$$

Cone Clutch Mechanics II

Torque Capacity Equation

$$T = \mu R_N R_m = \frac{\mu W R_m}{\sin \alpha}$$

Self-Locking Prevention

To allow smooth disengagement without jamming, the semi-cone angle must satisfy:

$$\alpha > \phi \quad \text{(where } \phi = \tan^{-1}\mu \text{ is the friction angle)}$$

In practice, α is chosen between 12.5° and 15°.

Centrifugal Clutches & Friction Materials I

Centrifugal Clutch Mechanics

- Operating speed ω: Centrifugal force on each shoe of mass m is $F_c = m\omega^2 r$.
- Spring pre-tension force F_s holds shoes inward until threshold speed ω_1 is reached.
- Net normal contact force against outer drum (R_{drum}):

$$F_N = F_c - F_s = m\omega^2 r - m\omega_1^2 r$$

- Total torque transmitted by n shoes:

$$T = n \cdot \mu \cdot (F_c - F_s) \cdot R_{\text{drum}}$$

Centrifugal Clutches & Friction Materials II

Friction Material Requirements & Selection

- **Organic Facings (Asbestos-free resin-bonded)**: High μ $(0.35 - 0.45)$, quiet operation; used in light automotive.
- **Sintered Metallic / Ceramics**: Excellent thermal stability $(> 300^\circ C)$; used in heavy machinery and racing.
- **Paper-based Composites**: High porosity, optimized for wet multi-plate automatic transmissions.

Lecture 24 Overview: Clutches I

Course Context: GTU DI03000141

- **Unit III**: Bearings, Clutches, Brakes, and Dynamometers.
- **Topic**: 3.3 Clutches — Mechanical principles, kinematic behavior, and force analysis of power transmission clutches.

Lecture 24 Overview: Clutches II

Learning Objectives

- Understand the fundamental role and operational requirements of friction clutches.
- Classify friction clutches based on geometry, mechanical construction, and actuation.
- Formulate mathematical models under **Uniform Pressure Theory (UPT)** and **Uniform Wear Theory (UWT)**.
- Derive torque transmission equations for Single-Plate, Multi-Plate, Cone, and Centrifugal Clutches.
- Analyze thermal behavior, friction lining properties, and design safety factors.

Functional Requirements of Friction Clutches I

Definition (Clutch)

A **clutch** is a mechanical device used to connect or disconnect a driving shaft and a driven shaft at the operator's will, enabling smooth transmission of power without stopping the prime mover.

Functional Requirements of Friction Clutches II

Essential Functional Requirements

1. **Gradual Engagement**: Must engage smoothly without shock or jerk to protect gears, shafts, and prime mover.
2. **High Torque Capacity**: Requires an optimal coefficient of friction (μ) and adequate axial force (W) to transmit maximum engine torque ($T = \mu W R_{eff}$).
3. **Heat Dissipation**: Friction during slip converts mechanical energy into thermal energy ($E_g = \int T_f \Delta\omega \, dt$). High thermal dissipation capability is critical to prevent thermal distortion and lining failure.
4. **Low Rotational Inertia**: Driven parts must possess minimum mass moment of inertia to facilitate rapid gear shifts without gear clashing.
5. **Dynamic Balance**: Rotating components must be dynamically balanced at high operational speeds.

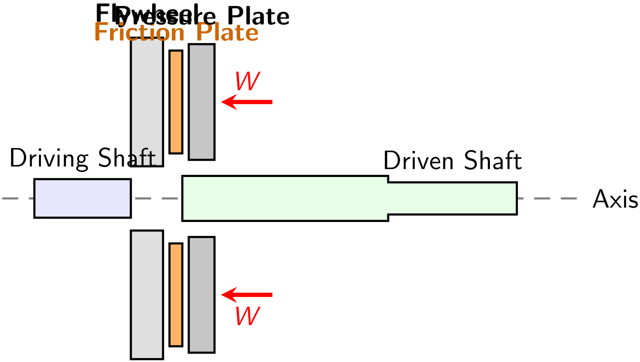
Classification of Friction Clutches I

Categorization by Construction Geometry

- **Plate/Disc Clutches** (Single-Plate & Multi-Plate): Axial pressure acts on flat annular friction surfaces.
- **Cone Clutches**: Conical mating surfaces produce a wedging action to multiply normal pressure.
- **Centrifugal Clutches**: Centrifugal force on sliding shoes automatically engages the clutch at elevated rotational speeds.

Parameter	Single Plate	Multi-Plate	Cone Clutch	Centrifugal
Torque Capacity	Moderate	High	High	Speed-Dependent
Radial Space	Large	Compact	Large	Moderate
Heat Dissipation	Excellent	Moderate/Poor	Moderate	Good
Actuation	Manual/Pedal	Manual/Pedal	Manual	Automatic
Applications	Automobiles	Motorcycles	Machine Tools	Mopeds, Saws

Table: Comparative Matrix of Friction Clutches


Flywheel
Pressure Plate
Friction Plate
W
Driving Shaft
Driven Shaft
Axis
W

Single Plate Clutch Anatomy II

Main Components & Interaction

- **Flywheel**: Bolted directly to the crankshaft (driving member).
- **Friction Plate**: Splined to the driven shaft, allowing axial freedom for engagement.
- **Pressure Plate & Springs**: Provides axial force W to clamp friction plate against flywheel.

Uniform Pressure Theory (UPT)

Assumptions & Governing Principle

- Assumes contact pressure p is uniform across the entire friction surface: $p = C = \text{constant}$.
- Applicable primarily to **new, perfectly flat, un-worn clutches**.

Mathematical Derivation

Consider an elemental annular ring of radius r and thickness dr ($r_2 \leq r \leq r_1$):

$$dA = 2\pi r \, dr$$

Total axial force W:

$$W = \int_{r_2}^{r_1} p \cdot 2\pi r \, dr = 2\pi p \int_{r_2}^{r_1} r \, dr = \pi p(r_1^2 - r_2^2) \implies p = \frac{W}{\pi(r_1^2 - r_2^2)}$$

Frictional torque on elemental ring dT:

$$dT = \mu \cdot (p \, dA) \cdot r = 2\pi\mu p r^2 \, dr$$

Integrating from r_2 to r_1 yields total torque capacity per contact surface pair:

$$T = 2\pi\mu p \int_{r_2}^{r_1} r^2 \, dr = \frac{2}{3}\pi\mu p(r_1^3 - r_2^3)$$

Substituting $p = \frac{W}{\pi(r_1^2 - r_2^2)}$:

$$T = \frac{2}{3}\mu W \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) = \mu W R_p \quad \text{where} \quad R_p = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$

Uniform Wear Theory (UWT) I

Assumptions & Wear Criterion

- Wear rate is proportional to frictional work, i.e., product of pressure and rubbing velocity: Wear $\propto p \cdot v \propto p \cdot r = C =$ constant.
- Represents the operational state of **worn/old clutches** and provides a conservative design criterion.
- Maximum pressure $p_{\max}$ occurs at inner radius r_2: $C = p_{\max} r_2$. Minimum pressure $p_{\min}$ occurs at r_1: $p_{\min} r_1 = C$.

Uniform Wear Theory (UWT) II

Mathematical Derivation

Total axial force W:

$$W = \int_{r_2}^{r_1} p \cdot 2\pi r \, dr = 2\pi C \int_{r_2}^{r_1} dr = 2\pi C(r_1 - r_2) = 2\pi p_{\max} r_2 (r_1 - r_2)$$

Frictional torque on elemental ring dT:

$$dT = \mu \cdot (p \cdot 2\pi r \, dr) \cdot r = 2\pi\mu C r \, dr$$

Integrating from r_2 to r_1:

$$T = 2\pi\mu C \int_{r_2}^{r_1} r \, dr = \pi\mu C(r_1^2 - r_2^2)$$

Substituting $C = \frac{W}{2\pi(r_1 - r_2)}$:

$$T = \mu W \left(\frac{r_1 + r_2}{2} \right) = \mu W R_w \quad \text{where} \quad R_w = \frac{r_1 + r_2}{2}$$

Multi-Plate Clutches Analysis I

Motivation & Operation

- When space constraints limit outer radius r_1, multiple friction discs are used to increase total contact area.
- Alternate plates are splined to driving and driven shafts.

Number of Contacting Surfaces

If n_1 = number of driving plates and n_2 = number of driven plates:

$$n = n_1 + n_2 - 1$$

Total torque transmitted by n friction pairs:

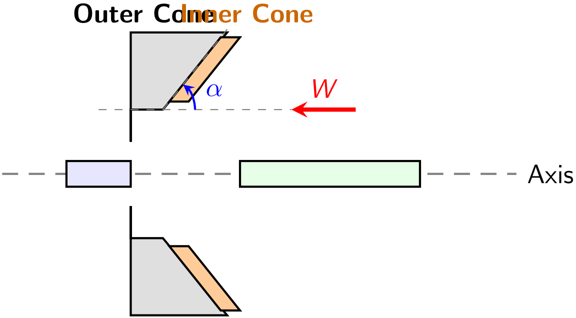
$$T = n \cdot \mu \cdot W \cdot R_{eff}$$

where $R_{eff} = R_p$ under UPT or $R_{eff} = R_w$ under UWT.

Design Comparison

Since $R_w < R_p$, Uniform Wear Theory yields a smaller torque rating than Uniform Pressure Theory. Engineers always design clutches using **UWT** to ensure adequate torque transmission throughout the component's lifespan.

Cone Clutch Anatomy and Mechanics I



Cone Clutch Anatomy and Mechanics II

Key Parameters & Wedging Action

- Semi-cone angle α: Angle of cone surface relative to axis of rotation (typically 12.5° to 15°).
- Wedging action increases normal reaction force $R_n = \frac{W}{\sin\alpha}$, multiplying effective torque.
- **Self-Locking Condition**: If $\alpha \leq \tan^{-1}(\mu)$, the clutch will self-lock, making disengagement difficult.

Cone Clutch Torque Capacity Analysis

Derivation of Torque Capacity

Elemental normal reaction dR_n on a conical strip of slant length $ds = \frac{dr}{\sin\alpha}$:

$$dR_n = p \cdot dA = p \cdot (2\pi r\, ds) = p \cdot 2\pi r \frac{dr}{\sin\alpha}$$

Axial load component $dW = dR_n \sin\alpha = p \cdot 2\pi r\, dr \implies W = \int_{r_2}^{r_1} p \cdot 2\pi r\, dr$.

Frictional torque on element $dT = \mu \cdot dR_n \cdot r = \frac{\mu p \cdot 2\pi r^2\, dr}{\sin\alpha}$.

- **Under Uniform Pressure Theory (UPT)**:

$$T = \frac{2}{3}\frac{\mu W}{\sin\alpha}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$$

- **Under Uniform Wear Theory (UWT)**:

$$T = \frac{\mu W}{\sin\alpha}\left(\frac{r_1 + r_2}{2}\right)$$

Axial Disengagement Force

Axial force required to disengage a cone clutch considering friction:

$$W_n = R_n(\mu\cos\alpha - \sin\alpha)$$

Centrifugal Clutch Working and Design I

Working Principle

- Operates on centrifugal force; automatically engages when driving speed exceeds a designated threshold speed ω_1.
- Commonly used in mopeds, chainsaws, and lawnmowers for automatic drive engage/disengage.

Centrifugal Clutch Working and Design II

Force Equilibrium on Sliding Shoes

Consider a centrifugal clutch with n shoes of mass m, with center of gravity at radius r_g:

- Centrifugal force at running speed ω: $P_c = m\omega^2 r_g$.
- Retracting spring force (at engagement threshold speed ω_1): $P_s = m\omega_1^2 r_g$.
- Net normal force exerted by each shoe against drum inner radius R:

$$P_n = P_c - P_s = mr_g(\omega^2 - \omega_1^2)$$

Total Torque Capacity

Total torque capacity for n shoes:

$$T = n \cdot \mu \cdot P_n \cdot R = n \cdot \mu \cdot mr_g(\omega^2 - \omega_1^2)R$$

Design Materials and Thermal Aspects I

Friction Lining Requirements

- High and uniform coefficient of friction ($\mu \approx 0.25 - 0.50$).
- Low wear rate, high resistance to thermal degradation (fade), and high thermal conductivity.

Common Materials

- **Molded Organic (Asbestos-free)**: Smooth operation, low noise, general automotive use.
- **Sintered Metal / Cerametallic**: High torque capacity, withstands heavy thermal loads.
- **Cork / Paper Composites**: Used in wet multi-plate motorcycle clutches operating in oil bath.

Thermal Considerations & Service Factor

- Heat generated during engagement period t_s:

$$E_g = \int_0^{t_s} T_f(\omega_{\text{driving}} - \omega_{\text{driven}})\, dt$$

- Temperature rise $\Delta T = \frac{E_g}{mc}$ must remain within safe operational limits of lining material.
- **Service Factor (k_s)**: Design torque rating $T_{\text{design}} = k_s \cdot T_{\text{engine}}$, where $k_s \approx 1.5 - 2.5$.

Summary and Key Formula Matrix I

Clutch Type	Uniform Pressure Theory (UPT)	Uniform Wear Theory (UWT)
Single Plate	$T = \frac{2}{3}\mu W \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$	$T = \mu W \left(\frac{r_1 + r_2}{2}\right)$
Multi-Plate (n pairs)	$T = n \cdot \frac{2}{3}\mu W \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$	$T = n \cdot \mu W \left(\frac{r_1 + r_2}{2}\right)$
Cone Clutch	$T = \frac{2}{3}\frac{\mu W}{\sin\alpha} \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$	$T = \frac{\mu W}{\sin\alpha} \left(\frac{r_1 + r_2}{2}\right)$
Centrifugal Clutch	$T = n \cdot \mu m r_g (\omega^2 - \omega_1^2) R$	

Table: Torque Capacity Summary Matrix for Friction Clutches

Summary and Key Formula Matrix II

Key Takeaways for GTU Examinations

1. UPT assumes $p = C$ (new clutches), while UWT assumes $p \cdot r = C$ (worn clutches).
2. UWT yields lower torque capacity ($R_w < R_p$) and is always preferred for conservative design.
3. In Multi-Plate clutches, total active surface pairs $n = n_1 + n_2 - 1$.
4. Cone clutch amplifies torque capacity by $1/\sin\alpha$, but semi-cone angle must satisfy $\alpha > \tan^{-1}(\mu)$ to prevent self-locking.

Lecture Overview: Brakes & Dynamometers I

Course & Topic Context

- **Course**: Theory of Machines and Mechanisms (GTU Course Code: DI03000141)
- **Unit**: Bearings, Clutches, Brakes & Dynamometers
- **Topic**: 3.4 Brakes and Dynamometers

Lecture Overview: Brakes & Dynamometers II

Key Learning Objectives

1. Differentiate between mechanical braking action (energy absorption/dissipation) and dynamometry (power measurement).
2. Formulate governing force and torque equations for single block (shoe) brakes under varying lever fulcrum positions.
3. Analyze belt friction mechanics in simple and differential band brakes.
4. Master concepts of self-energizing and self-locking conditions in lever brakes.
5. Classify dynamometers into absorption and transmission types, and apply analytical expressions for Prony, Rope brake, and Torsion dynamometers.

Brakes vs. Dynamometers: Fundamental Concepts I

Definitions & Principles

- **Brake**: A mechanical device applied to a moving machine member to bring it to rest, retard its motion, or prevent its motion by applying artificial frictional resistance.
  - Converts kinetic or potential energy of the moving system into thermal energy (heat).
  - The generated heat must be effectively dissipated into the surrounding atmosphere.
- **Dynamometer**: A measurement instrument used to determine the frictional resistance, operating torque, and brake power (B.P.) generated by a prime mover (engine, motor, turbine).
  - Absorbs or transmits mechanical power while simultaneously providing torque measuring parameters.

Brakes vs. Dynamometers: Fundamental Concepts II

Table: Comparative Analysis: Brakes vs. Dynamometers

Parameter	**Brake**	**Dynamometer**
Primary Objective	Retard, stop, or hold a moving mass	Measure output torque and shaft power
Energy Role	Absorbs and dissipates energy exclusively	Absorbs energy OR transmits energy
Core Measurement	Stopping distance / holding torque	Brake Power (B.P. $= \frac{2\pi NT}{60}$)
Typical Applications	Automobiles, trains, elevators, cranes	Engine test beds, motor characterization

Classification & Taxonomy of Frictional Brakes

Structural Classification

Brakes are broadly classified based on the direction of the acting force relative to the rotating drum and the friction element used:

- **Radial Brakes**: The braking force acts radially perpendicular to the axis of the rotating drum.
  - *Single Block or Shoe Brake*: Lever-operated single block pressed on drum rim.
  - *Double Block Brake*: Opposing blocks eliminating side thrust on shaft bearings.
  - *Band Brake*: Flexible metallic band lined with friction material wrapped around the drum.
  - *Internal Expanding Shoe Brake*: Internal shoes expanding outwards against drum interior (common in automobiles).
- **Axial Brakes**: The braking force acts parallel to the axis of the rotating shaft.
  - *Disc Brakes*: Caliper pads compressing axially against a rotating disc rotor.
  - *Cone Brakes*: Frictional engagement along conical surfaces.

Table: Classification Matrix of Industrial Brakes

Brake Category	Force Direction	Contact Geometry	Primary Advantage
Single Block	Radial	External Rim Sector	Mechanical simplicity, low cost
Double Block	Radial	Opposite Rim Sectors	Zero net bending load on shaft
Band Brake	Radial	Flexible Circumference	High wrap angle (θ), large torque
Internal Expanding	Radial	Internal Cylindrical Rim	High torque density, dust protected
Disc Brake	Axial	Parallel Flat Ring Faces	Excellent heat dissipation, anti-fade

Mathematical Analysis of Single Block Brake

Fundamental Braking Torque Equation

For a brake drum of radius r rotating under a block subjected to normal reaction R_N:

$$T_b = \mu \cdot R_N \cdot r$$

where μ is the coefficient of friction between block and drum rim.

Lever Equilibrium Cases (Moments about Fulcrum O_1)

Let P be the effort applied at distance l from shoe center, and x be the horizontal distance from fulcrum O_1 to shoe center.

- **Case 1: Line of action of friction force passes through fulcrum O_1**:

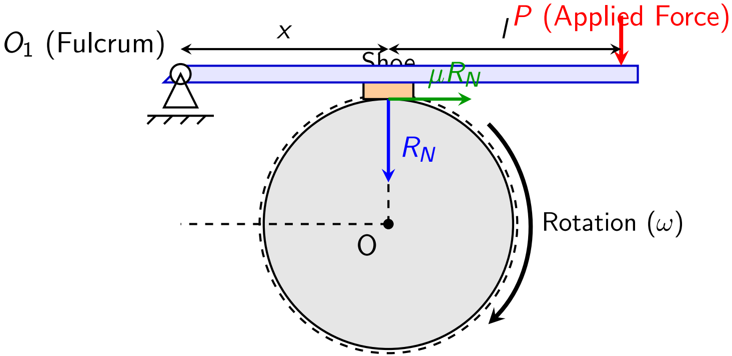
$$P \cdot (x + l) - R_N \cdot x = 0 \implies R_N = \frac{P(x + l)}{x}$$

- **Case 2: Line of action passes below fulcrum by distance a (Clockwise rotation)**:

$$P \cdot (x + l) - R_N \cdot x + \mu R_N \cdot a = 0 \implies R_N = \frac{P(x + l)}{x - \mu a}$$

  - **Self-Energizing Brake**: When frictional force assists in applying the brake (reduces required effort P).
  - **Self-Locking Condition**: If $x \leq \mu a$, then $P \leq 0$, meaning the brake engages automatically without external effort.

Schematic & Vector Forces of Single Block Brake I



Schematic & Vector Forces of Single Block Brake II

Diagram Key Highlights

- O: Center of rotating brake drum of radius r.
- O_1: Pivot fulcrum of the operating lever.
- R_N: Normal reaction force acting perpendicular to the contact surface.
- μR_N: Tangential friction force opposing drum rotation.

Band Brakes: Governing Equations & Tension Ratios I

Working Principle

A band brake consists of a flexible metal band lined with friction material wrapped around a drum of radius r. Braking action is achieved by tightening the band against the drum surface.

Band Brakes: Governing Equations & Tension Ratios II

Governing Equations

- **Braking Torque**:

$$T_b = (T_1 - T_2) \cdot r$$

where T_1 is tension in the tight side, and T_2 is tension in the slack side.

- **Limiting Ratio of Tensions** (Belt Friction Formula):

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

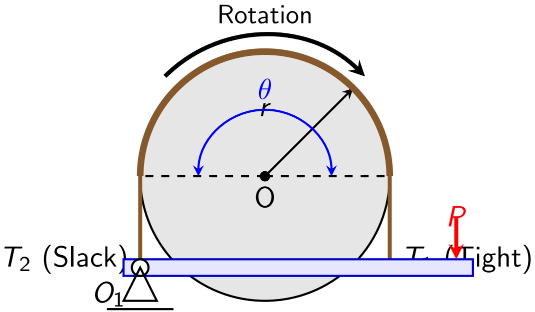
where θ is the angle of lap (wrap angle) in radians, and μ is the coefficient of friction.

Band Brakes: Governing Equations & Tension Ratios III

Classification of Band Brakes

- **Simple Band Brake**: One end of the band is attached directly to the fulcrum O_1, while the other end connects to the lever arm.
- **Differential Band Brake**: Neither end of the band is attached to the fulcrum; both ends are connected to the lever at different arm lengths (b_1 and b_2) relative to O_1. Can be engineered to be self-energizing or self-locking.

Schematic of Simple Band Brake Configuration I



Operating Mechanics

- For clockwise drum rotation, the tension ratio is governed by $\frac{T_1}{T_2} = e^{\mu\theta}$.
- Taking moments about fulcrum O_1: $P \cdot l = T_1 \cdot b$ (where b is the perpendicular distance from tight side T_1 to O_1).

Dynamometers: Fundamental Principles & Classification I

Role of Dynamometers

A dynamometer measures shaft torque (T) and rotational speed (N in RPM) to compute the Brake Power (B.P.) delivered by prime movers:

$$\text{B.P.} = \frac{2\pi NT}{60} \quad \text{[Watts]}$$

Dynamometers: Fundamental Principles & Classification II

Classification Categories

1. **Absorption Dynamometers**:
    - Entire mechanical power output of the prime mover is absorbed within the dynamometer and dissipated as heat.
    - Requires liquid or air cooling systems to handle heat load.
    - Examples: Prony Brake, Rope Brake, Hydraulic, and Eddy Current Dynamometers.
2. **Transmission Dynamometers**:
    - Mechanical energy is transmitted through the dynamometer to an external load machine without thermal absorption.
    - Torque is measured via structural elastic deflection or gear reaction forces.
    - Examples: Epicyclic-train, Belt Transmission, and Torsion Dynamometers.

Absorption Dynamometers: Prony & Rope Brakes I

Prony Brake Dynamometer

- Consists of friction wooden blocks clamped around a pulley mounted on the engine shaft.
- Frictional torque is balanced by dead weight W attached to a lever arm of length L.
- **Torque**: $T = W \cdot L$
- **Brake Power**: B.P. $= \frac{2\pi NWL}{60}$

Absorption Dynamometers: Prony & Rope Brakes II

Rope Brake Dynamometer

- Consists of flexible ropes wrapped 1 to 3 turns around a water-cooled engine flywheel.
- Top end is attached to a spring balance (reading S); bottom end carries dead weight W.
- **Effective Radius**: $R_{eff} = \frac{D+d}{2}$ (where $D =$ flywheel diameter, $d =$ rope diameter).
- **Net Frictional Force**: $F = (W - S)$
- **Torque**: $T = (W - S) \cdot \frac{D+d}{2}$
- **Brake Power**: B.P. $= \frac{\pi N(W-S)(D+d)}{60}$

Transmission and Torsion Dynamometers I

Epicyclic-Train Dynamometer

- Utilizes an epicyclic gear arrangement where spur gears transmit power from driving to driven shaft.
- Torque reaction on the gear carrier arm is balanced by dead weights on a calibrated scale arm.
- Eliminates continuous heat dissipation problems typical of absorption brakes.

Transmission and Torsion Dynamometers II

Torsion Dynamometer

- Measures the angular twist (θ) produced over a calibrated length (L) of a rotating shaft.
- Governed by the Torsion Equation:

$$\frac{T}{J} = \frac{G \cdot \theta}{L} \implies T = \frac{G \cdot J \cdot \theta}{L}$$

  where $J = \frac{\pi}{32} d^4$ is the polar moment of inertia, and G is shaft shear modulus.
- Widely used for high-power marine propulsion shafts and turbine test cells using optical or strain-gauge telemetry.

Summary & Comparative Formula Reference I

Table: Master Formula Summary for Brakes and Dynamometers

Device Type	**Governing Formula**	**Key Operational Conditions**
Single Block Brake	$T_b = \mu R_N r$	R_N depends on fulcrum offset a and rotation direction
Simple Band Brake	$T_b = (T_1 - T_2)r$, $\frac{T_1}{T_2} = e^{\mu\theta}$	θ in radians; tight side tension T_1 depends on rotation
Prony Brake	$T = W \cdot L$, B.P. $= \frac{2\pi NWL}{60}$	Direct absorption via wooden block friction
Rope Brake	$T = (W - S)\frac{D+d}{2}$	Requires internal flywheel water cooling
Torsion Dynamometer	$T = \frac{GJ\theta}{L}$, $J = \frac{\pi d^4}{32}$	Transmission type; measures elastic angular twist θ

Summary & Comparative Formula Reference II

Key Takeaways

- Mechanical brakes rely on friction to convert kinetic energy into heat; self-locking condition ($P \leq 0$) must be avoided in standard braking systems.
- Dynamometers enable precise power determination (B.P.); absorption types dissipate energy thermally, while transmission types measure torque non-destructively.

Lecture Overview & Learning Objectives I

- **Course**: Theory of Machines and Mechanisms (GTU Course Code: DI03000141)
- **Unit**: Bearings, Clutches, Brake & Dynamometer
- **Topic**: 3.4 Brakes and Dynamometers
- **Key Learning Objectives**:
  - Understand the fundamental mechanical principles of braking systems and kinetic energy transformation.
  - Derive expressions for braking torque in single block, band, and internal expanding shoe brakes.
  - Evaluate self-energizing and self-locking conditions in mechanical brakes.
  - Distinguish between absorption and transmission dynamometers.
  - Calculate brake power (BP) using various dynamometers (Prony brake, rope brake, torsion dynamometer).

Definition of a Brake

A mechanical device used to bring a moving system to rest, retard its motion, or hold it in a state of rest by applying artificial frictional resistance against moving elements.

- **Frictional Mechanism**: Contact between a stationary member (block, shoe, band, pad) and a rotating member (drum, disc).
- **Energy Transformation**:
  - Translating body kinetic energy: $E_k = \frac{1}{2}mv^2$
  - Rotating body kinetic energy: $E_k = \frac{1}{2}I\omega^2$
  - Dissipated heat energy: $Q = m_c c_p \Delta T$
- **Design Criteria**:
  - Required braking torque capacity (T_B).
  - Thermal dissipation capacity to prevent overheating and fading.
  - High wear resistance and stable coefficient of friction (μ).

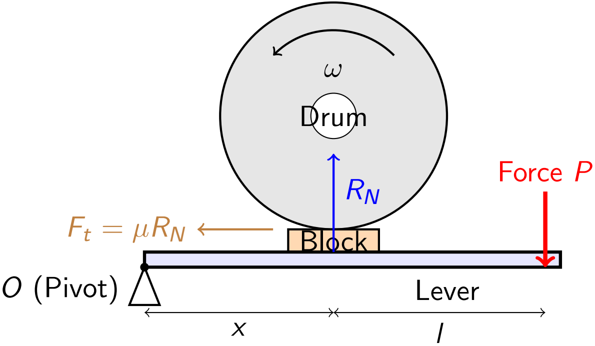
Introduction to Brakes II

- **Applications**: Automobiles, industrial hoists, elevators, railway stock, wind turbines.

- **Working Principle**: Consists of a friction block pressed radially against a rotating drum by an operating lever force P.
- **Braking Torque**:

$$T_B = F_t \cdot r = \mu R_N r$$

where r is the drum radius, μ is the coefficient of friction, R_N is the normal reaction, and F_t is the tangential friction force.
- **Effect of Drum Rotation**:
  - Depending on pivot location relative to the line of action of F_t, friction can either assist or oppose the applied force P.
- **Self-Energizing Brake**: Frictional force assists in applying the brake, reducing the required operator force P.
- **Self-Locking Brake**: Frictional moment is sufficient to engage and maintain braking action without external force ($P \leq 0$).



- **Equilibrium of Lever**: Taking moments about pivot O:

$$\sum M_O = 0 \implies P \cdot (x + l) + F_t \cdot h = R_N \cdot x$$

(for counter-clockwise drum rotation where friction assists).

Schematic & Analysis of Single Block Brake II

- **Self-Locking Condition**: Occurs when $x \leq \mu h$, resulting in $P \leq 0$.

Band Brakes and Belt Tension Ratio I

- **Construction**: A flexible steel band lined with friction material wrapped around a rotating brake drum.
- **Fundamental Tension Ratio Equation**:

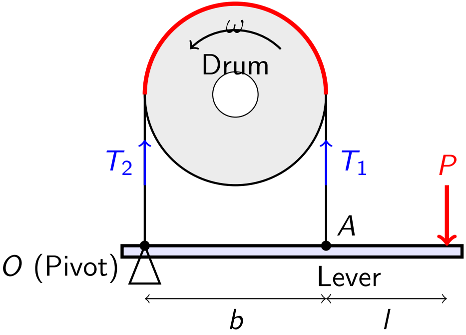
$$\frac{T_1}{T_2} = e^{\mu\theta}$$

  where:
  - T_1 = Tight side tension (N)
  - T_2 = Slack side tension (N)
  - μ = Coefficient of friction
  - θ = Wrap angle in radians
- **Net Braking Torque**:

$$T_B = (T_1 - T_2) \cdot r$$

- **Differential Band Brakes**: Both ends of the band are attached to different points on the operating lever relative to the pivot, enabling self-locking or high mechanical advantage.

Schematic Analysis of Simple Band Brake I



- **Lever Moment Balance**: Taking moments about pivot O:

$$P \cdot (b + l) = T_1 \cdot b$$

- **Directional Sensitivity**: Reversing drum rotation interchanges tight (T_1) and slack (T_2) sides, altering required operator effort P.

Internal Expanding Shoe Brakes I

- **Automotive Applications**: Widely used in drum brake systems of motor vehicles (rear wheels).
- **Key Components**: Rigid backplate, two curved shoes with friction linings, anchor pin, actuating cam / hydraulic wheel cylinder, retracting spring.
- **Leading vs. Trailing Shoe**:
  - **Leading (Primary) Shoe**: Friction force assists the actuating force (self-energizing), providing higher braking torque.
  - **Trailing (Secondary) Shoe**: Friction force opposes the actuating force (self-deenergizing).
- **Pressure Distribution**: Intensification of pressure along shoe lining is sinusoidal:

$$p = p_{\max} \sin \theta$$

- **Total Braking Torque**: Integration of normal and friction moments around anchor pin for both shoes.

Introduction to Dynamometers I

Definition & Function

A dynamometer is a device used to measure torque and rotational speed (RPM) of an engine, motor, or rotating prime mover to determine its shaft power.

- **Power Calculation**:

$$BP = T \cdot \omega = \frac{2\pi NT}{60} \quad \text{[Watts]}$$

  where T is shaft torque (N·m) and N is rotational speed (RPM).

- **Classification**:
  1. **Absorption Dynamometers**: Absorb and convert full engine power into heat via friction or fluid resistance.

Introduction to Dynamometers II

2. **Transmission Dynamometers**: Measure transmitted power without absorbing or converting it to heat.

- **Thermal Management**: Absorption types require liquid cooling systems during continuous performance testing.

Absorption Dynamometers I

- **Prony Brake Dynamometer**:
  - Mechanical friction blocks clamped on the shaft lever arm.
  - Torque: $T = W \cdot L$, where W is net weight and L is arm length.
- **Rope Brake Dynamometer**:
  - Rope wrapped around a water-cooled flywheel drum with dead weight W and spring balance S.
  - Effective torque:

$$T = (W - S) \cdot \frac{D + d}{2}$$

    where D is flywheel diameter and d is rope diameter.
- **Hydraulic Dynamometer**: Fluid friction / viscous resistance; power absorption capacity scales as N^3.
- **Eddy Current Dynamometer**: Electromagnetic induction creates retarding torque; offers precise electronic dynamic control.

Transmission Dynamometers I

- **Torsion Dynamometer**:
  - Measures elastic angular twist (θ) of shaft across length L:

$$\theta = \frac{TL}{GJ} \implies T = \frac{\theta GJ}{L}$$

  - Angle θ measured via optical sensors or strain gauges.
- **Epicyclic Gear Dynamometer**:
  - Uses a gear differential train; torque calculated by measuring reaction force required to hold carrier arm fixed.
- **Belt Transmission Dynamometer**:
  - Detects tension differential between tight and slack belt sides using intermediate force transducers.
- **Core Advantage**: Enables in-situ continuous efficiency monitoring without stopping power delivery.

Comparison: Brakes vs. Dynamometers I

Table: Comparative Matrix of Power Measurement and Retardation Devices

Parameter / Type	Mechanical Brakes	Absorption Dynamometers	Transmission Dynamometers
Primary Purpose	Stop or retard motion of machinery	Measure torque and power output	Measure torque without absorbing power
Energy Dissipation	100% dissipated as heat	100% absorbed and converted to heat	Negligible loss (power transmitted to load)
Key Elements	Blocks, drums, bands, pads	Prony block, rope-flywheel, water brake	Strain gauges, torsion bar, epicyclic gear
Measurement Method	Not calibrated for power measurement	Balance weight / force gauge ($W \cdot L$)	Angle of twist (θ) or reaction torque
Typical Application	Vehicles, cranes, elevators	Engine testing labs	Marine shafts, field monitoring

Summary of Key Formulas & Governing Principles I

Essential Equations for Examinations

- **Block Brake Torque**: $T_B = \mu R_N r$
- **Band Tension Ratio**: $\frac{T_1}{T_2} = e^{\mu\theta}$
- **Band Brake Torque**: $T_B = (T_1 - T_2) \cdot r$
- **Rope Brake Net Torque**: $T = (W - S) \cdot \frac{D+d}{2}$
- **Brake Power (BP)**: $BP = \frac{2\pi NT}{60} \quad [\text{W}]$
- **Torsion Shaft Twist**: $\theta = \frac{TL}{GJ}$

- **Self-Energizing vs. Self-Locking**:
  - Self-energizing reduces required actuation effort P.
  - Self-locking occurs when $P \leq 0$, holding the drum unconditionally without external effort.

Lecture 27: Overview & Learning Objectives I

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit:** Power Transmission Drives
- **Lecture 27:** 4.1 Concept and Types of Power Transmission Drives
- **Department:** Department of Mechanical Engineering

Lecture 27: Overview & Learning Objectives II

Learning Objectives

- Understand the fundamental role of power transmission systems between prime movers and driven machinery.
- Classify mechanical drives based on mode of transmission (friction vs. meshing), shaft distance, and velocity ratio.
- Differentiate between flexible and rigid transmission elements.
- Analyze the kinematics of belt, gear, chain, and rope drives including slip, creep, and line of action.
- Compare key performance parameters across different mechanical drives for engineering selection.

Introduction to Power Transmission Systems I

Role of Power Transmission

- Connects prime movers (electric motors, internal combustion engines, steam/gas turbines) to driven equipment (machine tools, conveyors, pumps, compressors).
- Prime movers typically operate at high rotational speeds with low torque outputs.
- Driven machinery often requires lower rotational speeds, higher torque, variable speed ratios, or altered direction/axis of motion.
- Transmission systems bridge this gap by converting and regulating speed, torque, and direction of motion.

Introduction to Power Transmission Systems II

Primary Functions of Transmission Drives

1. **Speed Matching:** Speed reduction or step-up ($\omega_1 \rightarrow \omega_2$).
2. **Torque Transformation:** Matching load torque requirements under power balance ($P = T_1\omega_1\eta = T_2\omega_2$).
3. **Direction & Axis Conversion:** Transferring power between parallel, intersecting, or non-parallel/non-intersecting shafts.

Introduction to Power Transmission Systems III

Major Media for Power Transmission

- **Mechanical Drives:** Belts, chains, gears, ropes, clutches, and couplings.
- **Electrical Drives:** Variable frequency drives and electric motors.
- **Hydraulic Drives:** Hydrostatic pumps/actuators and hydrodynamic fluid couplings.
- **Pneumatic Drives:** Compressed air systems for linear/rotary motion.

Classification of Mechanical Drives I

1. Classification by Mode of Transmission

- **Transmission by Friction:** Power is transmitted via friction forces developed between contact surfaces (e.g., flat/V-belt drives, rope drives, friction clutches, friction wheels).
- **Transmission by Meshing:** Power is transmitted via positive interlocking teeth or links (e.g., spur/helical/bevel gears, roller chain drives, toothed timing belts).

Classification of Mechanical Drives II

2. Classification by Center Distance (Shaft Distance)

- **Short Distance:** Gear drives (direct contact, compact layouts).
- **Medium Distance:** Chain drives (sprockets connected by flexible chain links).
- **Long Distance:** Belt and rope drives (continuous flexible connectors).

3. Classification by Velocity Ratio Capability

- **Positive Drives:** Maintain an exact and unvarying angular velocity ratio ($N_1/N_2 =$ constant) without slip (gears, timing belts, chain drives).
- **Non-Positive Drives:** Subject to friction slip and creep, resulting in slight velocity ratio variations (flat belts, V-belts, fiber ropes).

Flexible vs. Rigid Transmission Elements I

Rigid Transmission Elements (e.g., Gear Drives)

- Direct physical contact between mating rigid bodies without intermediate flexible connectors.
- **Advantages:** Extremely compact, constant velocity ratio, very high efficiency ($96\% - 99\%$), high load capacity.
- **Limitations:** Requires precise shaft alignment, continuous lubrication, low shock-absorption capability.

Flexible vs. Rigid Transmission Elements II

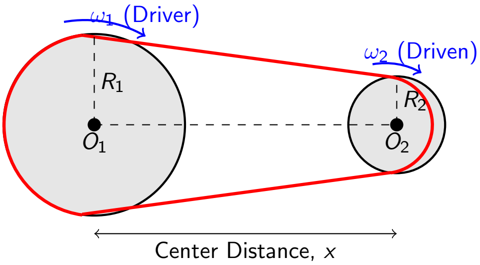
Flexible Transmission Elements (e.g., Belts, Chains, Ropes)

- Utilize an intermediate flexible continuous loop to bridge driver and driven shafts.
- **Advantages:** Can span large center distances, absorb shock loads, damp mechanical vibrations, tolerate minor shaft misalignments, protect machinery via overload slip.
- **Limitations:** Larger space requirement; non-positive flexible drives exhibit velocity ratio fluctuations due to elastic slip and creep under load.

Schematic of an Open Belt Drive I

Open Belt Drive Layout

Connects parallel shafts rotating in the **same direction**. Driver pulley of radius R_1 drives a driven pulley of radius R_2 over a center distance x.



Schematic of an Open Belt Drive II

Kinematic & Operational Highlights

- Ideal Velocity Ratio: $\frac{N_2}{N_1} = \frac{D_1}{D_2}$ (neglecting belt thickness and slip).
- The slack side is positioned on top to allow gravitational sag to increase the wrap angle θ, maximizing transmissible power.

Belt Drives: Kinematics, Slip, and Creep I

Frictional Belt Mechanics

- Limiting ratio of belt tensions for a flat belt: $\frac{T_1}{T_2} = e^{\mu\theta}$ where T_1 is tight-side tension, T_2 is slack-side tension, μ is coefficient of friction, and θ is contact angle (radians).
- Transmitted Power: $P = (T_1 - T_2)v$, where $v = \frac{\pi D_1 N_1}{60}$.

Phenomenon of Slip

Direct sliding of belt over pulley surface when load torque exceeds frictional resistance limit, reducing driven shaft speed.

Phenomenon of Creep

Relative micro-motion caused by elastic contraction and expansion of belt material as it passes from the tight side (T_1) to slack side (T_2).

Modified Velocity Ratio Formula

$\frac{N_2}{N_1} = \frac{D_1+t}{D_2+t}\left(1 - \frac{s}{100}\right)$ where t is belt thickness and s is total percentage slip ($s = s_1 + s_2$).

Gear Drives: Kinematics and Characteristics I

Fundamental Nature of Gear Drives

- Positive, rigid transmission via sequential engagement of teeth.
- Maintains an exact, unvarying angular velocity ratio: $\frac{\omega_1}{\omega_2} = \frac{N_1}{N_2} = \frac{d_2}{d_1} = \frac{T_2}{T_1}$ where d_1, d_2 are pitch diameters and T_1, T_2 represent tooth counts.

Law of Gearing

- **Statement:** The common normal to the mating tooth profiles at their point of contact must always pass through a fixed point on the line of centers, known as the **Pitch Point (*P*)**.
- Guarantees a strictly constant angular velocity ratio throughout tooth contact.

Gear Drives: Kinematics and Characteristics II

Operational Performance

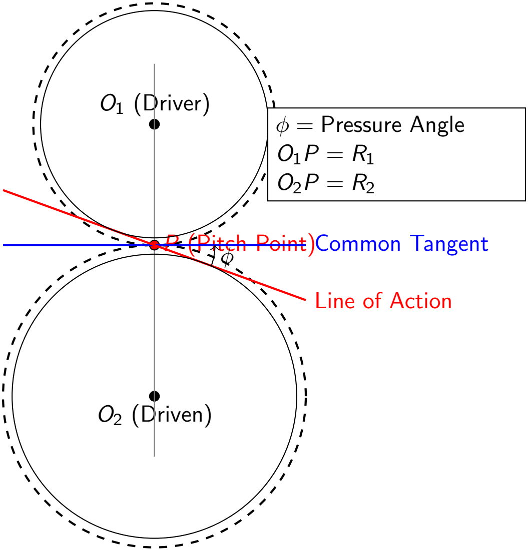
- High transmission efficiency ($96\% - 99\%$).
- Ideal for compact, high-torque, precise speed ratios across parallel, intersecting, or cross shafts.

Spur Gear Meshing and Line of Action I

Tooth Engagement Geometry

- **Pitch Circles:** Imaginary circles rolling without slip at Pitch Point P.
- **Line of Action (Pressure Line):** Path of force vector between teeth, tangent to base circles.
- **Pressure Angle (ϕ):** Angle between Line of Action and common tangent to pitch circles (typically 20° or 14.5°).

Spur Gear Meshing and Line of Action II



Spur Gear Meshing and Line of Action III

Force Decomposition

Tangential Force $F_t = F_n \cos\phi$ (transmits torque); Radial Force $F_r = F_n \sin\phi$ (loads shafts/bearings).

Chain and Rope Drives I

Chain Drives (Flexible Positive Drives)

- Endless articulated metal chains wrapping around toothed **sprockets**.
- Eliminates slip completely while accommodating medium center distances (1 m – 5 m).
- **Chordal Action (Polygon Effect):** Cyclic speed variation caused by chain links forming a polygon rather than a circle on the sprocket.

Rope Drives (Long Distance & Heavy Loads)

- **Fiber Ropes:** Made of synthetic or natural fibers; operate in grooved pulleys (wedge action) for long center distances.
- **Wire Ropes:** High-tensile steel wire strands wound around a core (steel/fiber).
- **Key Applications:** Elevators, cranes, mine hoists, cableways, heavy lifting machinery.
- **Key Features:** High load capacity, quiet operation, and built-in structural redundancy.

Performance Comparison Table

A multi-criteria performance comparison of major mechanical power transmission drives:

Parameter & **Belt Drive** & **Chain Drive** & **Gear Drive** & **Rope Drive**
Transmission Mode & Friction (non-positive) & Meshing (positive) & Meshing (positive) & Friction / Tension
Center Distance & Medium to Long & Medium (1 – 5 m) & Short (compact) & Very Long (> 5 m)
Velocity Ratio & Variable (slip/creep) & Exact constant & Exact constant & Slightly variable
Efficiency & 90% – 96% & 95% – 98% & 96% – 99% & 85% – 92%
Shock Absorption & Excellent & Fair & Poor & Good
Lubrication & None required & Required & Essential (continuous) & Wire lubrication
Space Required & Large & Medium & Compact & Large
Cost & Low & Moderate & High & Moderate

Table: Comparative performance metrics of mechanical drives.

Engineering Selection Criteria for Transmission Drives

Systematic Drive Selection Framework

Selecting the optimal drive system requires evaluating performance parameters against engineering and economic constraints.

Primary Selection Parameters

1. **Center Distance (x):**
   - Small x: Gear drives.
   - Medium x: Chain or V-belt drives.
   - Large x: Flat belt or rope drives.
2. **Velocity Ratio & Precision:**
   - Exact constant VR required $\implies$ Gear drive or timing belt / chain drive.
   - Non-critical VR allowed $\implies$ Flat or V-belt drive.
3. **Torque and Speed Requirements:**
   - High torque at low speeds $\implies$ Gear or chain drives.
   - High speeds at moderate power $\implies$ Belt drives.
4. **Operational Environment:**
   - Dust, extreme temperature $\implies$ Enclosed gear drives or metal chains.
   - Overload protection required $\implies$ Friction belt drives (slips upon jamming).

Lecture Overview: Power Transmission Drives I

Course Information

Course: Theory of Machines and Mechanisms (DI03000141)
Unit 4: Power Transmission Systems
Topic: 4.1 Concept and Types of Power Transmission Drives

Lecture Overview: Power Transmission Drives II

Key Learning Objectives

- Understand the fundamental need and functional role of mechanical power transmission in engineering machinery.
- Classify power transmission drives based on mode of transmission (friction vs. engagement) and intermediate linkage (flexible vs. rigid).
- Analyze velocity ratio, slip, creep, and tension dynamics in flexible drives (belts and chains).
- Study kinematic relationships, line of action, and fundamental law of gearing in rigid drives.
- Apply engineering selection criteria based on speed, torque, efficiency, center distance, and environmental conditions.

The Need for Power Transmission I

Mechanical power transmission drives transfer energy from a prime mover (source) to an output machine tool or actuator (sink), modifying speed, torque, and rotational direction.

The Need for Power Transmission II

Core Motivations and Functions

- **Source vs. Sink Matching:** Prime movers (electric motors, internal combustion engines) run efficiently at high, relatively constant speeds. Driven machines require lower speeds and higher torques.
- **Distance Span Adaptability:** Transmission spans range from fractions of a millimeter (precision watch mechanisms) to tens of meters (marine propeller drives).
- **Motion Transformation & Axis Realignment:** Converts rotary to linear motion, or connects parallel, intersecting, or non-parallel non-intersecting shafts.
- **Safety, Control & Protection:** Integrates clutches for intermittent engagement and safety shear pins or slip mechanisms for torque overload protection.

Classification of Mechanical Drives I

Mechanical energy transmission drives are categorized based on their underlying mode of motion transfer and physical connection.

Classification Criteria

1. **By Transmission Mode:**
   - *Friction Drives:* Belts, ropes, and friction clutches. Power transfer depends on frictional forces at contact surfaces.
   - *Mesh / Engagement Drives:* Gears and timing chains. Power transfer relies on direct mechanical intermeshing.
2. **By Distance & Intermediary Element:**
   - *Flexible Drives:* Belts, ropes, and chains using flexible intermediate links for medium to long center distances.
   - *Rigid / Direct Drives:* Gears and cam mechanisms for compact layouts with short center-to-center distances.

Classification of Mechanical Drives II

Positive vs. Non-Positive Drives

- **Non-Positive Drives (Friction):** Permit slip under overload, providing safety but yielding variable velocity ratios.
- **Positive Drives (Meshing):** Eliminate slip, maintaining an exact velocity ratio necessary for timing.

Flexible vs. Rigid Drives: Comprehensive Comparison

Selecting between flexible and rigid drives requires evaluating velocity ratio stability, shock absorption capacity, efficiency, and layout constraints.

Table: Comparison of Flexible and Rigid Power Transmission Drives

Parameter	**Flexible Drives (Belts/Chains)**	**Rigid Drives (Gears)**
Center Distance	Large to medium center distances	Short / compact center distances
Velocity Ratio	Variable (belts due to slip) / Exact (chains)	Exact and constant (positive drive)
Shock Absorption	Excellent elasticity dampens vibrations	Direct transmission (requires external dampers)
Efficiency	95% – 98% (friction/hysteresis losses)	Up to 99% per single mesh stage
Shaft Alignment	Primarily parallel shafts (crossed for non-parallel)	Parallel, intersecting, or non-parallel shafts
Maintenance	Periodic tensioning; low lubrication (belts)	Continuous casing and lubrication required

Concept of Velocity Ratio, Slip, and Creep I

The Velocity Ratio (VR) defines the kinematic relationship between driver and follower shafts. Frictional phenomena alter the theoretical performance in non-positive drives.

Theoretical Velocity Ratio

For driver pulley/gear 1 and follower 2:

$$VR_{theoretical} = \frac{N_1}{N_2} = \frac{d_2 + t}{d_1 + t}$$

where N is rotational speed in RPM, d is pitch diameter, and t is belt thickness.

Frictional Slip and Velocity Loss

Slip occurs when peripheral tension difference exceeds maximum friction capacity, causing relative motion between belt and pulley rim.

$$VR_{actual} = \frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}\left(1 - \frac{s}{100}\right)$$

where $s = s_1 + s_2$ is total percentage slip (s_1 between driver and belt, s_2 between belt and follower).

Concept of Velocity Ratio, Slip, and Creep III

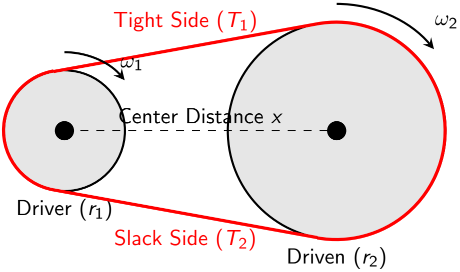
Belt Creep

Creep is the continuous elastic elongation and contraction of belt material as it transitions between high tension (T_1, tight side) and low tension (T_2, slack side), causing a $1\% - 2\%$ velocity loss:

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \cdot \left(\frac{E + \sqrt{\sigma_2}}{E + \sqrt{\sigma_1}} \right)$$

Flexible Drives: Flat and V-Belt Mechanics I

Belt drives rely on friction developed along the arc of contact to transmit tangential driving forces.



Flexible Drives: Flat and V-Belt Mechanics II

Governing Equations

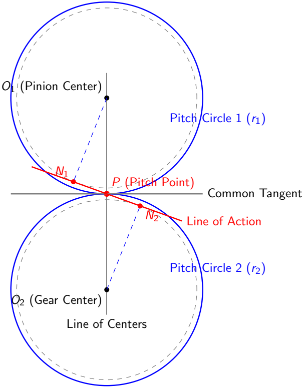
- **Limiting Tension Ratio (Flat Belt):** $\frac{T_1 - T_c}{T_2 - T_c} = e^{\mu\theta}$, where $T_c = mv^2$ is centrifugal tension, μ is coefficient of friction, and θ is contact angle (rad).
- **V-Belt Wedging Action:** Groove angle β (typically $30^\circ - 40^\circ$) enhances effective friction:

$$\mu' = \frac{\mu}{\sin(\beta/2)} \implies \frac{T_1 - T_c}{T_2 - T_c} = e^{\mu'\theta}$$

- **Transmitted Power:** $P = (T_1 - T_2)\, v$, where $v = \frac{\pi d N}{60}$.

Rigid Drives: Gear Kinematics and Line of Action

Gears transmit motion through direct positive contact along a defined pressure angle, ensuring exact kinematic ratios.



Fundamental Principles

- **Fundamental Law of Gearing:** Common normal at tooth contact must pass through pitch point P for constant angular velocity ratio $\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} = \frac{T_2}{T_1}$.
- **Line of Action & Pressure Angle (ϕ):** Normal line along which force is transmitted, inclined at ϕ (typically 20° or 25°) to common tangent.
- **Base Circle Radius:** $r_b = r \cos \phi$.

Chain Drives: Mechanics and Polygonal Effect I

Chain drives synthesize flexible belt center-distance advantages with positive gear transmission characteristics.

Key Operational Attributes

- **Positive Transmission:** Intermeshing roller links eliminate slip, maintaining an exact average velocity ratio.
- **Medium Center Distances:** Superior to compact gear arrangements for spanning intermediate center distances without heavy tension loss.
- **Lubrication Requirements:** Requires continuous fluid lubrication or oil bath casing to minimize pin-bush wear.

Chain Drives: Mechanics and Polygonal Effect II

Polygonal Effect (Chordal Action)

Because a chain sits on a sprocket as a polygon of pitch length p rather than a circle:

- Pitch radius fluctuates between $R_{max} = \frac{p}{2\sin(\pi/Z)}$ and $R_{min} = R_{max}\cos(\pi/Z)$, where Z is tooth count.
- Produces cyclic linear speed variation $\Delta v = v_{max} - v_{min}$, inducing high-frequency vibrations and noise.
- Mitigated by increasing the number of sprocket teeth ($Z \geq 17$ to 19).

Engineering Selection Criteria for Power Transmission Drives

Selecting the optimal drive system requires evaluating performance, operational constraints, and economic trade-offs.

Design Evaluation Matrix

1. **Power and Speed Regime:**
   - High torque, heavy loads $\implies$ Gear drives or heavy-duty duplex chains.
   - High rotational speed, moderate loads $\implies$ Flat or V-belts.
2. **Kinematic Precision:**
   - Synchronous timing required (e.g., engine valve timing) $\implies$ Timing belts, gears, or timing chains.
   - Non-critical precision, shock isolation required $\implies$ V-belts.
3. **Shaft Configuration:**
   - Parallel shafts $\implies$ Spur gears, helical gears, flat/V-belts, roller chains.
   - Intersecting shafts $\implies$ Bevel gears.
   - Non-parallel, non-intersecting shafts $\implies$ Worm gear drives, crossed helical gears.
4. **Environmental Resilience:** High temperatures, chemical exposure, or abrasives favor gearboxes or shielded chains over elastomeric belts.

Summary of Key Formulae and Design Principles

Summary Formulae Reference

- **Velocity Ratio (General):**

$$VR = \frac{N_1}{N_2} = \frac{d_2}{d_1} \quad (\text{Gears: } VR = \frac{T_2}{T_1})$$

- **Belt Velocity Ratio with Slip:**

$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}\left(1 - \frac{s}{100}\right)$$

- **Belt Tension Ratio (Flat vs. V-Belt):**

$$\text{Flat Belt: } \frac{T_1 - T_c}{T_2 - T_c} = e^{\mu\theta}, \qquad \text{V-Belt: } \frac{T_1 - T_c}{T_2 - T_c} = e^{\frac{\mu\theta}{\sin(\beta/2)}}$$

- **Centrifugal Tension & Belt Power:**

$$T_c = mv^2, \qquad P = (T_1 - T_2)\,v$$

- **Shaft Torque Power Relation:**

$$P = T\omega = \frac{2\pi NT}{60}$$

- **Base Circle Radius (Spur Gear):**

$$r_b = r\cos\phi$$

Course Overview & Drive Classification I

Introduction to Power Transmission Devices

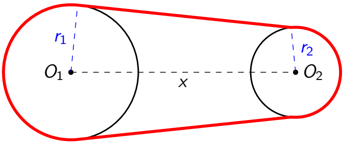
Flexible machine elements such as belts, ropes, and chains are widely used to transmit rotational power from one shaft to another over moderate to long center distances.

- **Classification Based on Belt Speed and Power Requirement:**
  - **Light Drives:** Belt speeds up to 10 m/s; used for transmitting small power (e.g., agricultural machinery, light machine tools).
  - **Medium Drives:** Belt speeds from 10 m/s to 22 m/s; used for medium power transmission (e.g., workshop machine tools).
  - **Heavy Drives:** Belt speeds exceeding 22 m/s; used for transmitting large power (e.g., industrial compressors, generators).
- **Classification Based on Shaft Arrangement:**

Course Overview & Drive Classification II

- **Open Belt Drive:** Used when shafts are parallel and rotate in the *same direction*.
- **Crossed Belt Drive:** Used when shafts are parallel and rotate in *opposite directions*.
- **Quarter-Turn Drive:** Used for shafts arranged at right angles (90°) to each other.

Geometry of Open Belt Drive



Geometric Parameters

Let r_1, r_2 be radii of driver and follower, and x be center distance.

- **Angle of Angle Shift (α):** $\sin\alpha = \frac{r_1 - r_2}{x}$
- **Angle of Wrap (Lap):**
  - Driver (larger pulley): $\theta_1 = \pi + 2\alpha$ rad
  - Follower (smaller pulley): $\theta_2 = \pi - 2\alpha$ rad
- **Total Length of Open Belt (L):** $L \approx 2x + \frac{\pi}{2}(d_1 + d_2) + \frac{(d_1 - d_2)^2}{4x}$

Velocity Ratio of Belt Drives I

Definition

The **velocity ratio** is the ratio of the angular speed of the driven pulley (follower) to that of the driving pulley (driver).

- **Ideal Velocity Ratio (Ignoring Belt Thickness):** V.R. $= \frac{N_2}{N_1} = \frac{d_1}{d_2}$ where N_1, N_2 are speeds in rpm, and d_1, d_2 are pulley pitch diameters.
- **Effect of Belt Thickness (t):** When thickness t is considered, effective pitch diameter becomes $(d + t)$: $\frac{N_2}{N_1} = \frac{d_1+t}{d_2+t}$
- **Compound Belt Drive Configuration:** For multiple pulleys keyed to intermediate shafts, the net velocity ratio is: $\frac{N_{\text{last}}}{N_{\text{first}}} = \frac{d_1 \times d_3 \times d_5 \times \dots}{d_2 \times d_4 \times d_6 \times \dots} = \frac{\text{Product of diameters of drivers}}{\text{Product of diameters of followers}}$

Phenomenon of Slip in Belt Drives I

Mechanism of Slip

Slip is the relative motion between the belt and pulley surface due to insufficient frictional grip under load.

- Let $s_1\%$ be slip between driver pulley and belt.
- Let $s_2\%$ be slip between belt and follower pulley.
- Linear velocity of belt (v_b):
  $v_b = v_1 \left(1 - \frac{s_1}{100}\right) = \frac{\pi(d_1+t)N_1}{60} \left(1 - \frac{s_1}{100}\right)$
- Linear velocity of follower pulley surface (v_2):
  $v_2 = v_b \left(1 - \frac{s_2}{100}\right) = \frac{\pi(d_2+t)N_2}{60}$
- **Velocity Ratio Considering Total Slip ($s = s_1 + s_2$):**
  $\frac{N_2}{N_1} = \frac{d_1+t}{d_2+t} \left(1 - \frac{s}{100}\right)$
- **Consequences:** Reduces power transmission efficiency, decreases velocity ratio, and causes thermal degradation of belt material.

Creep in Belts vs. Slip I

Elastic Creep Phenomenon

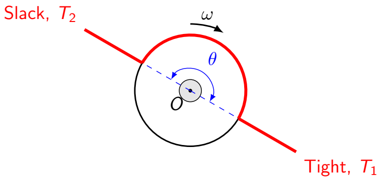
Creep is relative movement due to the continuous elastic stretch and contraction of belt material as it transitions between tight side (T_1) and slack side (T_2).

- **Velocity Ratio Considering Creep:** $\frac{N_2}{N_1} = \frac{d_1}{d_2}\left(\frac{E+\sqrt{\sigma_2}}{E+\sqrt{\sigma_1}}\right)$ where σ_1, σ_2 are stresses on tight and slack sides, and E is Young's Modulus of belt material.

Table: Comparison Between Slip and Creep in Belts

Parameter & **Slip** & **Creep**
Origin & Frictional capacity limit & Material elasticity property
Cause & Insufficient friction grip & Differential stretch ($T_1 > T_2$)
Occurrence & Intermittent / Under overload & Continuous during operation
Mitigation & Increase tension or μ & Increase Young's Modulus (E)

Belt Tension and Ratio of Tensions



Ratio of Driving Tensions

Deriving equilibrium of an elemental belt strip gives: $\frac{T_1}{T_2} = e^{\mu\theta}$ where:

- T_1 = Tension on tight side (N)
- T_2 = Tension on slack side (N)
- μ = Coefficient of friction
- θ = Angle of contact (rad)

Power Transmitted

$P = (T_1 - T_2) \cdot v$ [W] where $v = \frac{\pi d_1 N_1}{60}$ m/s.

Centrifugal Tension in Belts I

Effect of Centrifugal Force at High Speeds

When a belt rotates around a pulley at linear velocity v, centrifugal force creates additional tension T_c throughout the belt.

- **Centrifugal Tension Formula:** $T_c = mv^2$ where m is mass of belt per unit length (kg/m) and v is belt speed (m/s).
- **Total Tensions in Belt:**
  - Total tight side tension: $T_{t1} = T_1 + T_c$
  - Total slack side tension: $T_{t2} = T_2 + T_c$
- **Modified Ratio of Driving Tensions:** $\frac{T_{t1} - T_c}{T_{t2} - T_c} = \frac{T_1}{T_2} = e^{\mu\theta}$
- **Key Takeaway:** T_c increases peak belt stress (T_{t1}) without increasing power output, as $P = (T_{t1} - T_{t2})v = (T_1 - T_2)v$.

Condition for Maximum Power Transmission I

Optimization Problem

Let T be the maximum allowable safe tension in the belt. The tight side total tension is limited by $T_{t1} = T$.

- Tight side driving tension: $T_1 = T - T_c = T - mv^2$.
- Power transmitted equation: $P = (T_1 - T_2)v = T_1\left(1 - \frac{1}{e^{\mu\theta}}\right)v = C(Tv - mv^3)$ where $C = 1 - e^{-\mu\theta}$ is a constant.
- **Condition for Maximum Power ($\frac{dP}{dv} = 0$):** $\frac{d}{dv}(Tv - mv^3) = T - 3mv^2 = 0 \implies T_c = \frac{T}{3}$
- **Optimal Belt Speed for Maximum Power:** $v_{\text{opt}} = \sqrt{\frac{T}{3m}}$
- At maximum power, $T_1 = \frac{2}{3}T$ and $T_c = \frac{1}{3}T$.

V-Belt Drives and Groove Angle Effect I

Wedging Action in V-Belts

V-belts operate in grooved pulleys with included groove angle 2β (typically $30^\circ - 40^\circ$).

- **Virtual Coefficient of Friction (μ'):** Normal reaction from groove sides increases frictional grip: $\mu' = \frac{\mu}{\sin\beta} = \mu \csc\beta$
- **Modified Tension Ratio Equation for V-Belts:** $\frac{T_1}{T_2} = e^{\mu'\theta} = e^{\frac{\mu\theta}{\sin\beta}}$
- **Advantages over Flat Belts:**
  - Higher power capacity due to increased wedging friction.
  - Suitable for compact drives with short center distances.
  - Can achieve velocity ratios up to 7 : 1 or 10 : 1 without excessive slip.

Rope Drives: Fiber and Wire Ropes I

Application of Rope Drives

Rope drives are used for transmitting large power over long center distances (10 m to 30 m).

- **Fiber Ropes (Cotton, Hemp, Synthetic):**
  - Run in grooved pulleys with tension ratio: $\frac{T_1}{T_2} = e^{\mu\theta \csc \beta}$.
  - Smooth, quiet operation; suitable for textile mills and medium industrial drives.
- **Wire Ropes (Steel Wire Construction):**
  - Made of steel wires twisted into strands around a core (e.g., 6×19 or 6×37 designation).
  - Superior tensile strength, fatigue resistance, and durability.
  - **Applications:** Elevators, mine hoists, cranes, aerial ropeways.
  - **Bending Stress:** $\sigma_b = \frac{E \cdot d_w}{D}$ where d_w is wire diameter and D is drum diameter.

Comparative Summary and Key Equations I

Table: Comparison of Flat Belt, V-Belt, and Rope Drives

Feature & **Flat Belt** & **V-Belt** & **Rope Drive**
Center Distance & Medium (up to 8 m) & Short (Compact) & Long (up to 30 m)
Speed Ratio & 1 : 3 to 1 : 4 & Up to 1 : 7 & Up to 1 : 8
Tension Ratio & $e^{\mu\theta}$ & $e^{\mu\theta/\sin\beta}$ & $e^{\mu\theta\csc\beta}$
Slip Tendency & Higher & Negligible & Low
Power Capacity & Moderate & High & Very High

Summary of Critical Governing Formulas

- Centrifugal Tension: $T_c = mv^2$
- Max Power Condition: $T_c = \frac{T}{3} \implies v_{\text{opt}} = \sqrt{\frac{T}{3m}}$
- Power Transmitted: $P = (T_1 - T_2)v$

Course Introduction & Overview I

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Lecture 30:** 4.2 Belt and Rope Drives
- **Unit:** Power Transmission Systems and Mechanics
- **Department:** Department of Mechanical Engineering

Lecture Objectives

- Understand the mechanics of flexible connectors including flat belts, V-belts, and rope drives.
- Analyze velocity ratios, slip, creep, and tension ratios in power transmission.
- Evaluate the effect of centrifugal force, initial tension, and groove geometry on power transmission capacity.

Classification of Flexible Drives I

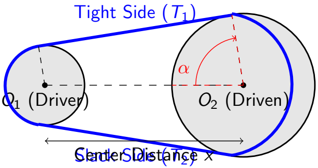
Flexible machine elements such as belts, ropes, and chains are used to transmit power over relatively long center distances. They absorb shock loads and damp vibrations, acting as safety devices against overload.

Classification of Flexible Drives II

Key Types of Flexible Drives

- **Flat Belts:** Used for moderate power transmission where center distance can be up to 8 meters; operates on flat pulleys.
- **V-Belts:** Cross-section is trapezoidal; operates in grooved pulleys (sheaves). Suitable for short center distances and high velocity ratios.
- **Ropes:** Made of fiber or steel wire. Fiber ropes are used for large center distances, while wire ropes are used for heavy hoisting (cranes, elevators).
- **Timing/Ribbed Belts:** Positive drive belts with teeth that mesh with sheave grooves, eliminating slip entirely.

Geometry of Open Belt Drive



- Open belt drive is used when pulleys rotate in the same direction.
- **Angle of wrap on smaller pulley:**

$$\theta = \pi - 2\alpha \quad \text{where } \sin\alpha = \frac{R-r}{x}$$

- **Angle of wrap on larger pulley:**

$$\theta = \pi + 2\alpha$$

- The power transmission capacity is governed by the smaller angle of wrap (driver pulley).

Velocity Ratio, Slip, and Creep

The speed ratio between the driver and driven pulleys is influenced by the physical characteristics of the belt, slip at the contact surfaces, and internal elastic creep.

Mathematical Formulation

- **Theoretical Velocity Ratio:**

$$\frac{N_2}{N_1} = \frac{d_1}{d_2}$$

- **Considering Belt Thickness (t):**

$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}$$

- **Effect of Slip ($s\% = s_1 + s_2$):**

$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}\left(1 - \frac{s}{100}\right)$$

- **Effect of Creep (c):**

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \times \frac{E + \sigma_2}{E + \sigma_1}$$

where E is Young's Modulus, and σ_1, σ_2 are stress levels on tight and slack sides.

Ratio of Belt Tensions in Flat Belts I

To transmit power, there must be a difference in tension between the tight side (T_1) and slack side (T_2). The limiting ratio of these tensions is determined by friction and contact wrap.

Limiting Tension Ratio

- **Governing Equation:**

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

  where μ is the coefficient of friction and θ is the angle of wrap on the smaller pulley (in radians).
- Assumes the belt is on the verge of slipping.

Torque and Power Transmitted

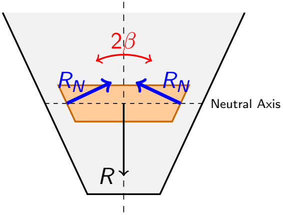
- **Torque Transmitted:**

$$T = (T_1 - T_2) \cdot r$$

- **Power Transmitted:**

$$P = (T_1 - T_2) \cdot v \quad \text{where } v = \frac{\pi d N}{60}$$

V-Belt Groove Mechanics



- V-belts operate in wedge-shaped grooves on pulleys, increasing normal reaction and friction force.
- Total reaction force:

$$R = 2R_N \sin\beta$$

  where 2β is the groove angle (typically 30° to 40°).
- Frictional force:

$$F = 2\mu R_N = \frac{\mu R}{\sin\beta} = \mu' R$$

- **Effective Coefficient of Friction:**

$$\mu' = \frac{\mu}{\sin\beta}$$

Tension Ratio in V-Belt and Rope Drives I

Governing Tension Equation

- Contact occurs along the inclined sides of the groove rather than the flat bottom.
- **Tension Ratio for V-Belt / Rope Drive:**

$$\frac{T_1}{T_2} = e^{\mu' \theta} = e^{\frac{\mu \theta}{\sin \beta}}$$

where θ is the angle of wrap on the smaller sheave and β is the groove half-angle.

Key Advantages of Wedging Action

- Since $\sin\beta < 1$ (typically $\beta \approx 15°$ to $20°$, so $\sin\beta \approx 0.25$ to 0.34), μ' is significantly higher than μ.
- Higher tension ratio allows smaller wrap angles and shorter center distances without slipping.

Centrifugal Tension & Max Power I

Centrifugal Tension (T_c)

- As the belt runs continuously over pulleys, centrifugal force produces additional tension:

$$T_c = m \cdot v^2$$

  where m is belt mass per unit length (kg/m) and v is belt velocity (m/s).
- Total tight side tension: $T_{t1} = T_1 + T_c$; total slack side tension: $T_{t2} = T_2 + T_c$.
- T_c acts equally on both sides, so effective driving force $(T_1 - T_2)$ remains unchanged, but total peak stress increases.

Centrifugal Tension & Max Power II

Condition for Maximum Power Transmission

- Maximum power occurs when centrifugal tension is one-third of maximum allowable tension ($T_{\max}$):

$$T_c = \frac{T_{\max}}{3}$$

- **Optimum Belt Velocity:**

$$v = \sqrt{\frac{T_{\max}}{3m}}$$

Initial Tension in Belt Drives I

Purpose of Initial Tension (T_0)

- When fitted onto pulleys, a belt is given initial tension T_0 to ensure sufficient friction grip and prevent slipping during starting.

Initial Tension in Belt Drives II

Formulations

- **Neglecting Centrifugal Tension (Hooke's Law assumption):**

$$T_0 = \frac{T_1 + T_2}{2}$$

- **Including Centrifugal Tension (T_c):**

$$T_0 = \frac{T_1 + T_2 + 2T_c}{2}$$

- **Barth's Equation (for non-linear elastic materials / ropes):**

$$T_0 = \frac{\sqrt{T_1} + \sqrt{T_2}}{2}$$

Comparison & Selection Criteria I

Table: Comparison of Flat Belts, V-Belts, and Rope Drives

Parameter	**Flat Belt**	**V-Belt**	**Rope Drive**
Center Distance	Large (up to 8 m)	Small / Compact	Very Large (up to 30–50 m)
Velocity Ratio	Moderate (up to 4:1)	High (up to 7:1)	High (up to 8:1)
Efficiency	High ($\approx$ 98%)	Moderate ($94 - -96\%$)	Moderate ($\approx$ 95%)
Slip & Creep	Susceptible to slip	Minimal due to wedge effect	Low slip in grooved sheaves
Applications	Machine tools, agricultural machines	Automotive engines, industrial drives	Cranes, elevators, heavy hoisting

Lecture 31: Chain Drives I

Course Details

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit 4:** Power Transmission Systems
- **Topic:** 4.3 Chain Drives
- Mechanical Engineering Curriculum

Introduction to Chain Drives I

A chain drive is a mechanical power transmission system that uses a flexible chain with links engaging with the teeth of sprockets to transfer rotational motion and torque between shafts.

Key Characteristics

- Combines the flexibility of belt drives with the positive engagement (no-slip) of gear drives.
- Maintains a constant velocity ratio since there is no slip or creep.
- Typically used for short to medium center distances where gears are impractical and belts would slip.
- Allows a single chain to drive multiple shafts simultaneously.

Comparison with Belt and Gear Drives I

Evaluating chain drives against alternative power transmission systems helps engineers choose the optimal drive for a given application.

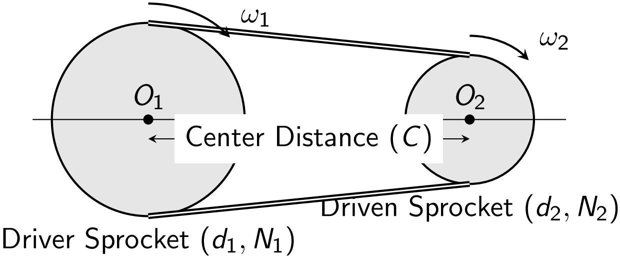
Table: Comparison of Mechanical Power Transmission Systems

Parameter	Belt Drive	Chain Drive	Gear Drive
Drive Type	Non-positive (Slip)	Positive (No slip)	Positive (No slip)
Center Distance	Large	Medium	Small / Short
Efficiency	92–96%	Up to 98%	98–99%
Shaft Loading	High (Initial tension)	Low (No initial tension)	Medium
Environment	Temperature limited	High temp / harsh ok	Enclosed / lubricated
Cost & Noise	Low cost, Quiet	Medium cost, Noisy	High cost, Moderate

Comparison with Belt and Gear Drives II

Summary of Advantages

- **High Efficiency:** Transmission efficiency up to 98%, superior to standard belt drives.
- **Shaft Loading:** Lower radial load than belt drives due to absence of initial tension.
- **Environmental Resilience:** Operates reliably at high temperatures and in abrasive or wet conditions.


ω_1
ω_2
O_1
O_2
Center Distance (C)
Driven Sprocket (d_2, N_2)
Driver Sprocket (d_1, N_1)

Chain Drive Geometry and Layout II

Geometrical Parameters

- **Sprocket Pitch Circle Diameter (d):** Given by $d = \frac{p}{\sin(180^\circ/N)}$, where p is the chain pitch.
- **Velocity Ratio (VR):** Expressed as $VR = \frac{N_2}{N_1} = \frac{d_2}{d_1} = \frac{n_1}{n_2}$.
- **Center Distance (C):** Recommended value is 30 to 50 times the chain pitch to prevent excessive slack.
- **Arc of Contact:** Minimum angle of wrap on the smaller sprocket should be at least 120°.

Classification of Chains I

Chains are categorized by function, load capacity, and construction into three main groups:

1. Hoisting and Hauling Chains

- Used for lifting loads at relatively low speeds.
- Examples include oval link crane chains and block chains.

2. Conveyor Chains

- Designed for continuous transport of materials.
- Feature attachments for buckets, brackets, or flight plates.

Classification of Chains II

3. Power Transmission Chains

- High-speed, high-efficiency chains engineered specifically to transmit torque.
- Subdivided into bush chains, roller chains, and silent (inverted-tooth) chains.

Roller Chain Construction I

A standard roller chain consists of alternating inner and outer links assembled to provide optimal strength, wear resistance, and flexibility.

Main Components

- **Link Plates (Outer & Inner):** Resist tensile loads; manufactured from high-tensile alloy steels and heat-treated for fatigue resistance.
- **Pins:** Pressed into outer plates, serving as journals. Must possess high shear strength and surface hardness.
- **Bushes:** Pressed into inner plates, acting as sleeve bearings for the pins. Hardened to withstand joint wear.
- **Rollers:** Rotate freely on bushes; roll over sprocket teeth to substitute rolling friction for sliding friction.

The Polygonal Effect (Chordal Action) I

The polygonal effect is a fundamental kinematic phenomenon in chain drives where the chain forms a polygon rather than a true circle around the sprocket, causing periodic velocity fluctuations.

The Polygonal Effect (Chordal Action) II

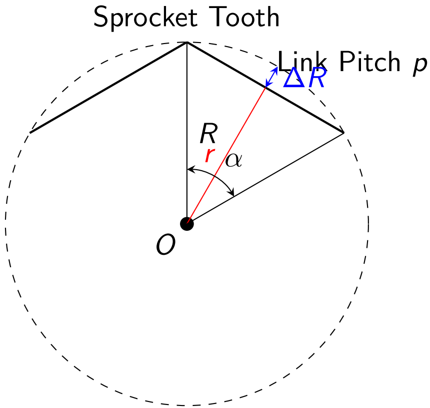
Kinematic Characteristics

- As the sprocket rotates, the effective operating radius oscillates between a maximum (R) and a minimum ($r = R\cos(\alpha/2)$).
- This variation leads to periodic cyclic acceleration and deceleration of the chain, inducing severe vibration.
- The percentage variation in linear chain velocity is defined by:

$$\frac{\Delta V}{V_{\text{mean}}} = \pi\left(\frac{1}{\cos(180^\circ/N)} - 1\right)$$

- Increasing the number of teeth N significantly mitigates this velocity variation (typically $N_1 \geq 17$ for power transmission).

Geometry of Chordal Action I



Geometry of Chordal Action II

Mathematical Relations

- **Maximum Pitch Radius:** $R = \frac{p}{2\sin(\alpha/2)}$, occurring when a link joint aligns with the sprocket center line.
- **Minimum Pitch Radius:** $r = R\cos(\alpha/2)$, occurring when the link midpoint is perpendicular to the center line.
- **Radial Fluctuation:** $\Delta R = R - r = R\,(1 - \cos(\alpha/2))$, driving torsional vibrations.
- **Subtended Angle:** $\alpha = \frac{360^\circ}{N}$, representing the angle subtended by a single pitch at the sprocket center.

Design and Selection Considerations I

Designing a reliable chain drive requires balancing mechanical power capacity, geometry, operating life, and noise constraints.

Key Design Rules

- **Minimum Teeth on Pinion:** $N_1 \geq 17$ for smooth operation at moderate speeds; $N_1 \geq 21$ for high-speed applications.
- **Pitch Selection:** Use the smallest practical pitch to minimize polygonal effect, mass, dynamic impact, and noise.
- **Multi-strand Chains:** Duplex or triplex chains are selected to increase power capacity while keeping pitch size small.
- **Lubrication Classes:**
  - *Class I:* Manual or drip lubrication (low speeds).
  - *Class II:* Oil bath or disc lubrication (medium speeds).
  - *Class III:* Oil stream / pressure spray lubrication (high speeds).

Failure Modes and Maintenance I

Chain drives operate under severe friction and cyclic load conditions; understanding failure mechanisms ensures timely maintenance.

Dominant Failure Modes

- **Fatigue Failures:** Cyclic tensile stress leads to fatigue cracks in link plates or fatigue failure of pins.
- **Wear-Induced Elongation:** Articulation wear between pins and bushes increases joint clearance, causing apparent chain pitch "stretch".
- **Galling:** Excessive speed or load under insufficient lubrication leads to localized micro-welding and pin surface tearing.
- **Sprocket Tooth Erosion:** Sliding contact between rollers and teeth wears out the sprocket profile, necessitating joint replacement of chain and sprockets.

Course Details

- **Course**: Theory of Machines and Mechanisms (GTU Code: DI03000141)
- **Unit**: Power Transmission
- **Topic**: 4.3 Chain Drives – Kinematics, Dynamics, and Design Principles
- **Target Audience**: Undergraduate Mechanical Engineering Students

Lecture 32: 4.3 Chain Drives – Overview II

Learning Objectives

1. Understand the fundamental working principles and classification of chain drives.
2. Analyze the kinematics of chain drives, specifically chordal action (polygonal effect).
3. Compute velocity ratios, linear chain speeds, pitch diameter, and required chain length.
4. Perform dynamic force analysis including tangential power force, centrifugal tension, and sag tension.
5. Identify critical failure modes and select appropriate lubrication systems.

Introduction to Chain Drives I

Working Principle

Chain drives represent an intermediate class of power transmission, combining the positive engagement of gear drives with the flexibility and long center-distance capability of belt drives.

- **Positive Drive Mechanism**: Unlike belts, there is no slippage or creep, ensuring a constant average velocity ratio.
- **Geometric Configuration**: Consists of an endless chain wrapped around two or more toothed wheels called sprockets.
- **Space and Efficiency**: Highly efficient (up to 98%), offering a compact design compared to belt drives of similar capacity.
- **Harsh Environment Operation**: Suitable for high temperatures, wet, or dusty conditions where belts would rapidly degrade.

Table: Comparison of Drive Characteristics

Parameter	**Belt Drive**	**Chain Drive**	**Gear Drive**
Velocity Ratio	Variable (slip)	Constant (average)	Constant (exact)
Center Distance	Large	Medium	Small
Efficiency	90–95%	96–98%	97–99%
Space Occupied	Large	Compact	Very Compact
Maintenance	Low	Medium (Lubrication)	High

Classification of Power Transmission Chains I

Chains are broadly classified into three main groups based on their application:

1. Hoisting and Hauling Chains

Used for lifting loads or dragging materials at low speeds (e.g., stud link chains, weldless chains, oval link chains).

2. Conveyor Chains

Designed for continuous material transport, featuring attachments for carrying buckets, flight bars, or slats.

Classification of Power Transmission Chains II

3. Power Transmission Chains

Designed for high-speed, high-power energy transmission between parallel shafts.

- **Roller Chains**: Most common, consisting of inner/outer link plates, pins, bushings, and rollers. Standardized under ANSI B29.1 and ISO 606.
- **Silent / Inverted Tooth Chains**: Uses inverted tooth link plates to provide quiet operation at high speeds with minimal chordal action.

Schematic of a Chain Drive System I

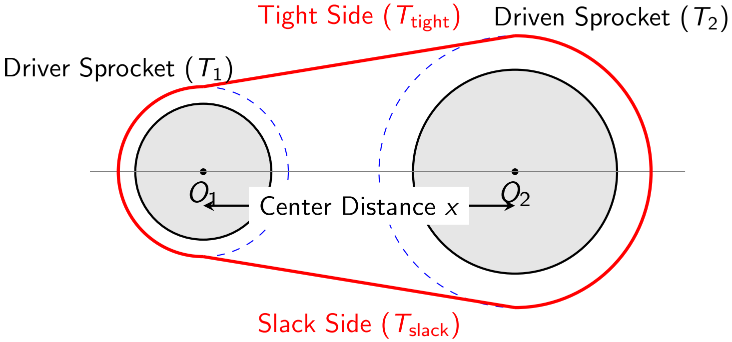


Figure: Schematic Layout of Sprocket and Chain System

Schematic of a Chain Drive System II

Key Geometric Terminology

- **Chain Pitch (*p*)**: Distance between centers of adjacent joint pins.
- **Pitch Circle Diameter (*D* or *PCD*)**: Theoretical circle passing through chain joint centers:

$$D = \frac{p}{\sin(180^\circ/T)}$$

- **Center Distance (*x*)**: Distance between driving and driven shaft centers.
- **Tight Side vs. Slack Side**: Driving strand carries tension; return strand carries minimal tension and self-weight sag.

Kinematics & Chordal Action I

The Polygonal Effect

A critical characteristic of chain drives is the **polygonal effect** (or *chordal action*), which causes periodic fluctuations in linear chain speed and instantaneous velocity ratio.

- **Nature of Engagement**: The chain wraps around the sprocket as a polygon of sides equal to pitch p, not a continuous smooth circle.
- **Fluctuating Effective Radius**: As the sprocket rotates at constant angular speed ω, the radius of the chain line varies between:
  - Pitch Circle Radius: $R = \frac{p}{2\sin(\pi/T)}$
  - Chordal Radius: $R_c = R\cos\left(\frac{\pi}{T}\right)$

Kinematics & Chordal Action II

- **Linear Velocity Fluctuations**:

$$v_{max} = \omega \cdot R$$
$$v_{min} = \omega \cdot R_c = \omega \cdot R \cos\left(\frac{180^\circ}{T}\right)$$

- **Velocity Variation Percentage**:

$$\frac{\Delta v}{v_{max}} = \frac{v_{max} - v_{min}}{v_{max}} = 1 - \cos\left(\frac{180^\circ}{T}\right)$$

- **Mitigation**: Use sprockets with at least 17 teeth ($T_1 \geq 17$, preferably $T_1 \geq 19$ for high speeds) to keep velocity variation under 1.5%.

Geometry of the Polygonal Effect I

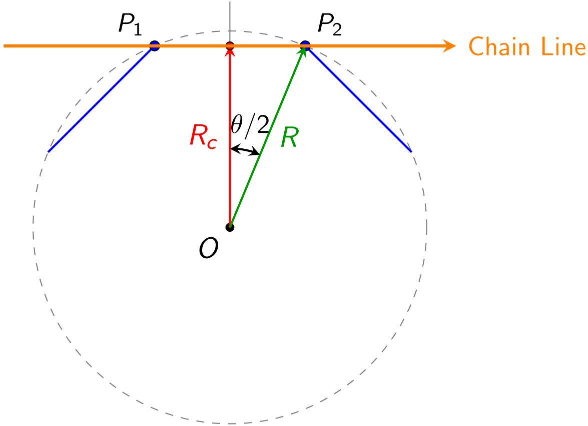


Figure: Geometry of Chordal Action on Sprocket Polygon

Geometry of the Polygonal Effect II

Kinematic Consequences

- **Impact & Noise**: Periodic acceleration of the chain mass generates impact stresses, noise, and chordal vibration.
- **Speed Variations**: Decreases rapidly as tooth count T increases (1.5% for $T = 19$, down to 0.5% for $T = 30$).

Velocity Ratio & Chain Length Calculations I

1. Average Velocity Ratio (i)

While instantaneous velocity ratio oscillates due to chordal action, the average velocity ratio remains exact:

$$i = \frac{N_1}{N_2} = \frac{T_2}{T_1}$$

where N_1, N_2 are rotational speeds in rpm, and T_1, T_2 are tooth counts of driver and driven sprockets.

2. Average Linear Chain Speed (v)

$$v = \frac{p \cdot T_1 \cdot N_1}{60000} \quad [\text{m/s}]$$

where p is chain pitch in mm.

Velocity Ratio & Chain Length Calculations II

3. Chain Length in Pitches (L_p)

The total chain length expressed in number of links/pitches L_p is given by:

$$L_p = 2\left(\frac{x}{p}\right) + \frac{T_1 + T_2}{2} + \frac{(T_2 - T_1)^2}{4\pi^2\left(\frac{x}{p}\right)}$$

Note: L_p must be rounded up to the nearest integer (preferably an even number to avoid offset links).

Velocity Ratio & Chain Length Calculations III

4. Corrected Center Distance (x)

After selecting an integral pitch count L_p, recalculate actual center distance x:

$$x = \frac{p}{4}\left[\left(L_p - \frac{T_1 + T_2}{2}\right) + \sqrt{\left(L_p - \frac{T_1 + T_2}{2}\right)^2 - \frac{8(T_2 - T_1)^2}{4\pi^2}}\right]$$

Force Analysis in Chain Drives

Unlike belt drives, chain drives run without initial tension ($T_0 = 0$).

Components of Chain Tension

1. **Tangential Power Transmitting Force (P_t)**:

$$P_t = \frac{P}{v} = \frac{1000 \times \text{Power (kW)}}{v} \quad [\text{N}]$$

2. **Centrifugal Tension (P_c)**:

$$P_c = m \cdot v^2 \quad [\text{N}]$$

   where m is chain mass per unit length (kg/m) and v is linear speed (m/s).

3. **Sag Tension (P_s)**: Tension due to gravity on the slack side strand:

$$P_s = k \cdot m \cdot g \cdot x \quad [\text{N}]$$

   where k is the catenary coefficient ($k = 2$–6 depending on drive layout and inclination).

Maximum Total Tension ($T_{\max}$)

$$T_{\max} = P_t + P_c + P_s$$

The design requirement ensures that the breaking strength W_B satisfies:

$$\text{Factor of Safety (FOS)} = \frac{W_B}{T_{\max}} \geq 7 \text{ to } 10$$

Failure Modes & Design Limits I

Chain drive components undergo severe cyclic mechanical stresses under operating conditions.

Table: Primary Failure Modes in Power Transmission Chains

Failure Mode	Mechanism / Cause	Operating Regime
Fatigue Failure	Cyclic tensile stress on link plates	Low to medium speeds
Impact Fatigue	Roller/bushing tooth impact	High speeds
Bush/Pin Wear	Joint friction leading to pitch elongation	Inadequate lubrication
Galling / Seizure	Metal transfer between pin and bush	High speed & high load
Sprocket Tooth Wear	Abrasive/adhesive wear of teeth	Unaligned sprockets

Failure Modes & Design Limits II

Design Considerations

- Maximum allowable pitch elongation due to wear is typically 1.5% to 3%.
- Beyond this limit, chain climbs sprocket teeth and causes ride-out or tooth breakage.

Proper lubrication reduces joint friction and can increase chain service life by up to 90%.

Lubrication Classes (ISO / ANSI Standards)

- **Class I (Manual / Drip Lubrication)**: Applied periodically by oil can or drip feed. Suitable for low speeds ($v < 4$ m/s).
- **Class II (Oil Bath / Slinger Disk)**: Lower strand dips into oil reservoir or slinger wheel splashes oil. Moderate speeds ($4 \text{ m/s} \leq v \leq 7$ m/s).
- **Class III (Oil Stream / Pressure Spray)**: Continuous pump-fed oil jet directed at inside of chain loop. High speeds ($v > 7$ m/s).

Lubrication & Maintenance II

Installation & Maintenance Practices

- **Sag Control**: Maintain slack-side sag at 2–4% of center distance x.
- **Alignment**: Ensure shaft parallelism and axial alignment within $\pm 0.05^\circ$.
- **Oil Selection**: Use non-detergent petroleum oils; viscosity tailored to temperature (SAE 20 to SAE 50).

Summary & Key Formulas I

Summary of Key Design Equations

- **Velocity Ratio**: $i = \frac{N_1}{N_2} = \frac{T_2}{T_1}$
- **Pitch Diameter**: $D = \frac{p}{\sin(180^\circ/T)}$
- **Linear Speed**: $v = \frac{p \cdot T_1 \cdot N_1}{60000}$
- **Polygonal Fluctuation**: $\frac{\Delta v}{v_{\max}} = 1 - \cos\left(\frac{180^\circ}{T}\right)$
- **Total Tension**: $T_{\max} = P_t + P_c + P_s$

Summary & Key Formulas II

Review Questions

1. Explain why a chain drive provides a constant average velocity ratio but not a constant instantaneous velocity ratio.
2. Derive the expression for velocity variation in a chain drive due to chordal action.
3. Why is an odd number of teeth preferred for sprockets and an even number of links for chains?

Lecture 33: 4.4 Gear Drives and Gear Trains I

Course Information

- **Course:** Theory of Machines and Mechanisms (GTU Course Code: DI03000141)
- **Department:** Department of Mechanical Engineering
- **Topic:** Gear Drives, Tooth Kinematics, and Multi-Shaft Gear Trains

Lecture Objectives

- Understand the fundamental kinematics of direct-contact positive gear drives.
- Derive and analyze the fundamental **Law of Gearing** for conjugate tooth action.
- Master gear tooth terminology including pitch circle, module, and circular pitch.
- Classify and calculate velocity ratios for Simple, Compound, and Reverted gear trains.
- Formulate and evaluate planetary gear systems using the **Tabular Method**.

Lecture 33: 4.4 Gear Drives and Gear Trains III

Prerequisites

- Power transmission systems, velocity ratio concepts, and friction drives.
- Instantaneous center of rotation and relative velocity analysis.

Introduction to Gear Drives I

Definition & Basic Mechanism

A **gear drive** is a direct contact mechanical transmission system that transfers power and motion between rotating shafts through the sequential meshing of teeth profiles.

Introduction to Gear Drives II

Key Characteristics of Gear Drives

- **Positive Drive Mechanism:** Eliminates slip (unlike belt or rope drives), maintaining an exact, constant angular velocity ratio ($\frac{\omega_1}{\omega_2} = \text{constant}$).
- **High Power Density & Compactness:** Enables high power and torque transmission across small center-to-center distances.
- **High Mechanical Efficiency:** Achieves mechanical efficiencies ranging from 97% to 99% per meshing pair under proper elastohydrodynamic lubrication.
- **Geometric Versatility:** Configurable for parallel, intersecting, and non-parallel non-intersecting (skew) shaft arrangements.

Introduction to Gear Drives III

Design Considerations

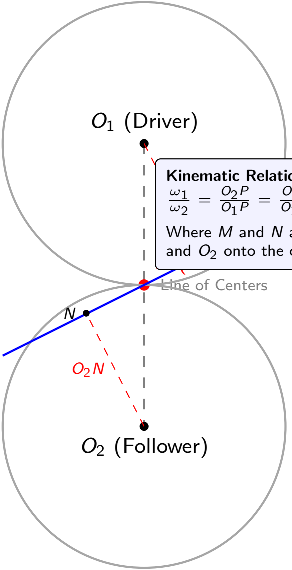
Requires high manufacturing precision, specialized lubrication systems, rigid shaft mountings, and dynamic load balancing at high rotational speeds.

The Law of Gearing I

Fundamental Statement

To maintain a **constant angular velocity ratio** between two mating gears, the common normal to the tooth profiles at the point of contact must always pass through a fixed point (the **Pitch Point**) on the line joining the centers of rotation.

The Law of Gearing II



Kinematic Relation:
$\frac{\omega_1}{\omega_2} = \frac{O_2P}{O_1P} = \frac{O_2N}{O_1M} = \text{constant}$

Where M and N are perpendicular projections from O_1 and O_2 onto the common normal.

The Law of Gearing III

Conjugate Tooth Profiles

Profiles that satisfy the Law of Gearing are termed **conjugate curves**. The most widely adopted conjugate profile in industrial applications is the **involute curve** due to its insensitivity to small center distance variations.

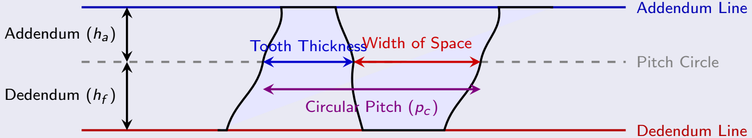
Classification of Gears

Classification by Shaft Alignment

1. **Parallel Shafts:**
    - **Spur Gears:** Straight teeth parallel to the axis of rotation; easy to manufacture, but higher operational noise.
    - **Helical Gears:** Teeth cut at a helix angle; smooth, gradual engagement, higher load capability, generates axial thrust.
    - **Rack & Pinion:** Convert rotational motion to linear motion (infinite radius pitch circle).
2. **Intersecting Shafts:**
    - **Straight Bevel Gears:** Conical pitch surfaces for power transmission at fixed angles (typically 90°).
    - **Spiral Bevel Gears:** Helical-style teeth on conical pitch surfaces; quieter operation and higher torque carrying capacity.
3. **Non-Parallel, Non-Intersecting Shafts (Skew Shafts):**
    - **Worm & Worm Wheel:** High speed reduction ratios ($10:1$ to $100:1+$) in a single stage; self-locking capability.
    - **Hypoid Gears:** Offset shaft axes widely utilized in automotive rear-axle differentials.

Gear Tooth Terminology I

Tooth Geometry Schematic



Gear Tooth Terminology II

Quantitative Definitions

- **Pitch Circle Diameter (d):** Diameter of the theoretical pure-rolling pitch circle.
- **Module (m):** Fundamental index of gear size ($m = \frac{d}{T}$, measured in mm).
- **Circular Pitch (p_c):** Arc distance along pitch circle ($p_c = \frac{\pi d}{T} = \pi m$).
- **Diametral Pitch (P_d):** Number of teeth per unit pitch diameter ($P_d = \frac{T}{d} = \frac{\pi}{p_c}$).
- **Addendum (h_a) & Dedendum (h_f):** Radial distance from pitch circle to tip ($h_a = 1m$) and root ($h_f = 1.25m$).

Simple Gear Trains I

Definition

A **simple gear train** is an assembly where each intermediate or terminal shaft carries exactly **one gear wheel**.

Kinematic Velocity Ratio

For a train of gears $1, 2, 3, \ldots, n$ mounted on separate parallel shafts:

$$\frac{N_1}{N_2} = -\frac{T_2}{T_1}, \quad \frac{N_2}{N_3} = -\frac{T_3}{T_2}, \quad \ldots, \quad \frac{N_{n-1}}{N_n} = -\frac{T_n}{T_{n-1}}$$

Multiplying these equations yields the overall speed ratio:

$$\text{Speed Ratio} = \frac{N_{\text{in}}}{N_{\text{out}}} = \frac{N_1}{N_n} = (-1)^{n-1} \cdot \frac{T_n}{T_1}$$

Simple Gear Trains II

Role of Idler Gears

- Intermediate gears (idlers) cancel out algebraically in the numerical speed ratio equation.
- **Primary Functions of Idlers:**
  1. Control the direction of rotation of the output shaft.
  2. Bridge large center-to-center distances without requiring excessively large gears.

Compound Gear Trains I

Definition & Architecture

A **compound gear train** features at least one intermediate shaft carrying **two or more gears** rigidly keyed together, so they rotate at identical angular velocities.

Compound Gear Trains II

Velocity Ratio Derivation

Consider driver gear 1 driving gear 2 on shaft B, while gear 3 (also on shaft B) drives gear 4 on shaft C:

$$N_2 = N_3 \implies \frac{N_1}{N_2} = \frac{T_2}{T_1} \quad \text{and} \quad \frac{N_3}{N_4} = \frac{T_4}{T_3}$$

Overall Speed Ratio:

$$\frac{N_{\text{in}}}{N_{\text{out}}} = \frac{N_1}{N_4} = \left(\frac{N_1}{N_2}\right) \times \left(\frac{N_3}{N_4}\right) = \frac{T_2 \cdot T_4}{T_1 \cdot T_3}$$

Generalizing for k meshing stages:

$$\text{Speed Ratio} = \frac{N_{\text{driver}}}{N_{\text{driven}}} = \frac{\prod \text{Teeth of Driven Gears}}{\prod \text{Teeth of Driver Gears}}$$

Compound Gear Trains III

Key Advantage

Achieves extremely high overall velocity ratios within a compact physical volume compared to a simple gear train.

Reverted Gear Trains I

Definition

A **reverted gear train** is a specialized compound gear train where the input shaft and output shaft are **co-axial** (collinear axes of rotation).

Reverted Gear Trains II

Geometric & Module Constraints

Let shaft 1 (input gear T_1) drive shaft 2 (compound gears T_2 and T_3), which drives shaft 3 (output gear T_4, co-axial with shaft 1):

- **Center Distance Constraint:**

$$r_1 + r_2 = r_3 + r_4 \implies \frac{d_1}{2} + \frac{d_2}{2} = \frac{d_3}{2} + \frac{d_4}{2}$$

- **Teeth Number Constraint (for uniform module m):** Since $d = m \cdot T$:

$$m_1(T_1 + T_2) = m_2(T_3 + T_4)$$

If both stages share the same module ($m_1 = m_2 = m$):

$$T_1 + T_2 = T_3 + T_4$$

Reverted Gear Trains III

Applications

Wristwatches and clocks (hour/minute hand co-axial drivers), lathe gearboxes, and co-axial industrial speed reducers.

Epicyclic (Planetary) Gear Trains I

Concept & Structure

An **epicyclic gear train** consists of gears whose axes of rotation move in space relative to the frame. One or more **planet gears** rotate about their own axes while revolving around a central **sun gear**, supported by a rotating **carrier arm**.

Core Components

- **Sun Gear (A):** Central gear located at the main axis of rotation.
- **Planet Gears (B):** Gears meshing with the sun gear and revolving around it.
- **Carrier / Arm (C):** Link holding the axes of the planet gears.
- **Ring / Annulus Gear (D):** Internal gear concentric with the sun gear.

Epicyclic (Planetary) Gear Trains II

Kinematic Characteristics

- **Degrees of Freedom (DOF):** Has 2 degrees of freedom. Two independent motion inputs (e.g., sun speed and arm speed) uniquely specify the output speed.
- **High Power Density:** Compact co-axial layout with torque split across multiple planet gears.

Analysis of Epicyclic Gear Trains I

The Tabular Method

The **Tabular Method** systematically separates carrier rotation from relative gear mesh rotation.

Table: Kinematic Analysis Table for Simple Epicyclic Gear Train

Step & **Condition of Motion** & **Arm** (C) & **Sun Gear** (A) & **Planet Gear** (B)
1 & Fix Arm C, give $+1$ rev to Sun A & 0 & $+1$ & $-\frac{T_A}{T_B}$
2 & Fix Arm C, give $+y$ revs to Sun A & 0 & $+y$ & $-y \cdot \frac{T_A}{T_B}$
3 & Add $+x$ revs to all elements & $+x$ & $+x$ & $+x$
4 & **Total Motion** & $+\mathbf{x}$ & $\mathbf{x} + \mathbf{y}$ & $\mathbf{x} - \mathbf{y} \cdot \frac{\mathbf{T_A}}{\mathbf{T_B}}$

Analysis of Epicyclic Gear Trains II

Application Procedure

1. Identify fixed or driven elements from problem specifications.
2. Set up boundary conditions using the Total Motion row (e.g., if Arm is fixed, $x = 0$; if Ring Gear is fixed, $x + y\frac{T_A}{T_D} = 0$).
3. Solve linear simultaneous algebraic equations for unknowns x and y.
4. Compute target component rotational speeds ($N = x + \text{offset}$).

Course & Lecture Overview: Gear Drives and Gear Trains I

Course Details

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit 4**: Power Transmission Devices
- **Topic**: Fundamentals of Toothed Gearing & Kinematic Analysis of Gear Trains
- **Target Audience**: GTU Mechanical Engineering Students (Semester 3)

Course & Lecture Overview: Gear Drives and Gear Trains II

Lecture Objectives

- Understand the fundamental physics of positive motion transmission using gear drives.
- Derive and apply the Law of Gearing and analyze involute tooth geometry.
- Calculate path of contact, arc of contact, and contact ratio.
- Evaluate gear tooth interference, undercutting, and minimum tooth count limits.
- Classify gear trains (Simple, Compound, Reverted, Epicyclic) and solve kinematic speed ratios.
- Master the Tabular Method for epicyclic gear train analysis and calculate torque/power equilibrium.

Introduction to Gear Drives & Law of Gearing I

Introduction to Gear Drives

Gears are toothed wheels that transmit motion and power between rotating shafts via direct successive tooth engagement. They represent a **positive drive mechanism** with exact velocity ratios, zero slip, and high mechanical efficiency.

Classification Based on Shaft Orientation

- **Parallel Shafts**: Spur gears (straight teeth), Helical gears (inclined teeth), Herringbone gears (double helical).
- **Intersecting Shafts**: Straight Bevel gears, Spiral Bevel gears.
- **Non-Parallel, Non-Intersecting Shafts**: Worm and Worm wheel, Skew bevel/Spiral gears.

Introduction to Gear Drives & Law of Gearing II

Fundamental Law of Gearing

The **Law of Gearing** states: *The common normal to the tooth profiles at the point of contact must always pass through a fixed point (pitch point P) on the line connecting the centers of rotation of the two mating gears.*

$$\frac{\omega_1}{\omega_2} = \frac{O_2P}{O_1P} = \frac{r_2}{r_1} = \frac{T_2}{T_1}$$

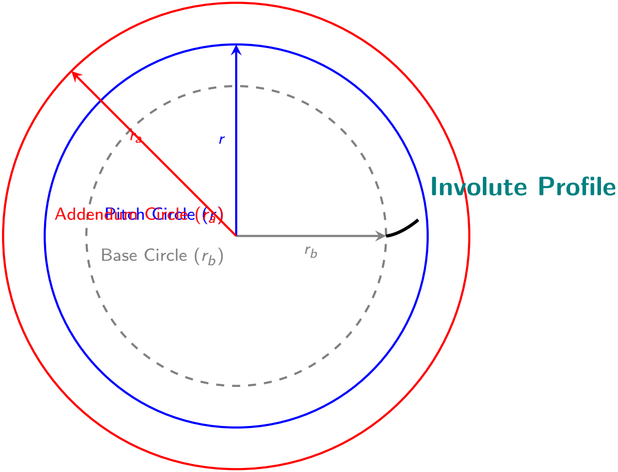
where ω is angular velocity, r is pitch circle radius, and T is the number of teeth.

Involute Gear Tooth Profile Geometry I

Involute Profile Definition

An **involute** is the curve traced by a point on a taut string unwinding from a stationary base circle. Involute profiles satisfy the Law of Gearing and maintain a constant velocity ratio even if center distance varies slightly.

Involute Gear Tooth Profile Geometry II



Involute Gear Tooth Profile Geometry III

Key Geometric Circles & Parameters

- **Base Circle (r_b)**: The circle from which the involute curve is generated ($r_b = r\cos\phi$).
- **Pitch Circle (r)**: Pure rolling reference circle where kinematic contact and module $m = d/T$ are defined.
- **Pressure Angle (ϕ)**: Angle between the line of action and common pitch tangent (standard 20°).
- **Addendum (a) & Dedendum (d_d)**: Radial distance above ($1.0m$) and below ($1.25m$) the pitch circle.

Path of Contact (L_c)

The line segment along the line of action where tooth contact begins and ends:

$$L_c = \text{Path of Approach} + \text{Path of Recess} = KP + PL$$

$$L_c = \sqrt{R_a^2 - R_b^2} + \sqrt{r_a^2 - r_b^2} - (R + r)\sin\phi$$

where R_a, r_a are addendum circle radii, and R_b, r_b are base circle radii.

Arc of Contact (L_a)

The distance traveled by a point on the pitch circle during complete engagement of a pair of teeth:

$$L_a = \frac{\text{Path of Contact}}{\cos\phi} = \frac{L_c}{\cos\phi}$$

Contact Ratio (CR)

The average number of tooth pairs in contact during power transmission:

$$CR = \frac{\text{Arc of Contact}}{\text{Circular Pitch}} = \frac{L_a}{p_c} = \frac{L_a}{\pi m}$$

- **Requirement**: $CR > 1.0$ for continuous motion (typically $1.4 \leq CR \leq 1.8$).

Interference and Undercutting in Involute Gears I

Mechanism of Interference

Interference occurs when the tip of a gear tooth contacts the flank of its mating pinion below the base circle, where the flank profile is non-involute. This causes digging and jamming during engagement.

Undercutting

When gears with low tooth counts are generated using rack cutters, the cutter removes material from the tooth flank root below the base circle to prevent interference. This weakens the tooth bending strength significantly.

Minimum Number of Teeth to Avoid Interference

- **Pinion mating with a Rack**:

$$t_{\min} = \frac{2a_w}{\sin^2 \phi}$$

For standard full-depth teeth $(a_w = 1, \phi = 20^\circ)$, $t_{\min} = 18$ teeth.

- **Pinion mating with a Gear of Gear Ratio** $G = T/t$:

$$t_{\min} = \frac{2a_w}{\sqrt{1 + G(G+2)\sin^2 \phi} - 1}$$

Methods to Eliminate Interference

- Increase pressure angle ϕ (e.g., from 14.5° to 20° or 25°).
- Use stub teeth (reducing addendum height a_w).
- Apply profile displacement / tooth modification (shift addendum).

Classification of Gear Trains I

What is a Gear Train?

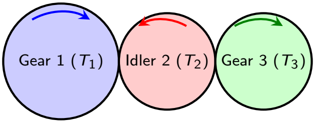
A gear train is a combination of two or more meshing gears mounted on rigid frames to transmit rotary motion and power between input and output shafts, achieving desired velocity and torque ratios.

Major Types of Gear Trains

1. **Simple Gear Train**: Each shaft carries only one gear wheel.
2. **Compound Gear Train**: At least one shaft carries multiple gears that rotate at the same angular speed.
3. **Reverted Gear Train**: The input and output shafts are co-axial (aligned along the same axis).
4. **Epicyclic (Planetary) Gear Train**: Axes of one or more gears rotate about the center axis of another gear.

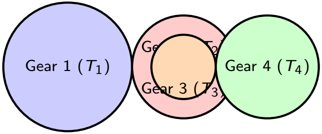
Kinematics of Simple and Compound Gear Trains I

Simple Gear Train



$$\frac{N_3}{N_1} = -\frac{T_1}{T_3}$$

Compound Gear Train



$$\frac{N_4}{N_1} = \frac{T_1 \cdot T_3}{T_2 \cdot T_4}$$

Kinematics of Simple and Compound Gear Trains II

Kinematic Velocity Ratio Equations

- **Simple Train**: Intermediate gears (idlers) affect only the direction of rotation, not the magnitude of speed ratio:

$$\text{Speed Ratio} = \frac{N_{\text{out}}}{N_{\text{in}}} = (-1)^{n-1}\frac{T_{\text{in}}}{T_{\text{out}}}$$

- **Compound Train**: Allows very large velocity ratios within compact spaces:

$$\text{Train Value} = \frac{N_{\text{last}}}{N_{\text{first}}} = \frac{\text{Product of Teeth on Drivers}}{\text{Product of Teeth on Driven}}$$

Reverted Gear Trains and Co-axial Shafts I

Reverted Gear Train Configuration

A reverted gear train is a special compound train in which the first driving shaft and the last driven shaft are co-axial. Used in clocks, lathe gearboxes, and industrial speed reducers.

Reverted Gear Trains and Co-axial Shafts II

Geometric & Kinematic Constraints

For gears 1, 2, 3, and 4 (where Gear 1 meshes with Gear 2, and Gear 3 on the same shaft as 2 meshes with Gear 4):

1. **Center Distance Equality**:

$$r_1 + r_2 = r_3 + r_4$$

   If all gears have the same module m:

$$T_1 + T_2 = T_3 + T_4$$

2. **Velocity Ratio**:

$$\frac{N_4}{N_1} = \frac{T_1 \cdot T_3}{T_2 \cdot T_4}$$

Analysis of Epicyclic Gear Trains: Tabular Method I

Epicyclic Motion Concept

An epicyclic gear train has gears whose axes move relative to the frame. It possesses 2 degrees of freedom, requiring two input speeds (or one speed + one fixed member) to fully specify kinematic state.

Step & **Operation** & **Arm (C)** & **Sun Gear (A)** & **Planet Gear (B)**
1 & Fix Arm, give $+1$ rev to Sun A & 0 & $+1$ & $-\frac{T_A}{T_B}$
2 & Fix Arm, give $+x$ rev to Sun A & 0 & $+x$ & $-x \cdot \frac{T_A}{T_B}$
3 & Add $+y$ rev to all elements & $+y$ & $+y$ & $+y$
4 & **Total Motion** & $+y$ & $x + y$ & $y - x \cdot \frac{T_A}{T_B}$

Table: Standard Tabular Method for Epicyclic Gear Train Kinematics

Analysis of Epicyclic Gear Trains: Tabular Method II

Procedure for Solving Epicyclic Problems

1. Set up total motion expressions for each gear component from Step 4.
2. Apply known boundary conditions (e.g., fixed ring gear: $N_R = 0$, input shaft speed N_A).
3. Solve linear algebraic equations for unknowns x and y.

Torque and Power Analysis in Epicyclic Trains I

Static Equilibrium Conditions

For an epicyclic train operating at constant speed, three external torques act on the system: Input torque (T_i), Output torque (T_o), and Fixing/Reaction torque (T_r).

$$T_i + T_o + T_r = 0 \quad \Longrightarrow \quad T_r = -(T_i + T_o)$$

Torque and Power Analysis in Epicyclic Trains II

Energy & Power Balance

Assuming 100% mechanical efficiency (ideal frictionless gear train):

$$P_i + P_o + P_r = 0 \quad \Longrightarrow \quad T_i\omega_i + T_o\omega_o + T_r\omega_r = 0$$

Since the reaction member is fixed ($\omega_r = 0$), power transmission satisfies:

$$T_i\omega_i + T_o\omega_o = 0 \quad \Longrightarrow \quad T_o = -T_i\left(\frac{\omega_i}{\omega_o}\right)$$

Fixing / Holding Torque Calculation

$$T_r = -T_i\left(1 - \frac{\omega_i}{\omega_o}\right) = T_i\left(\frac{\omega_i}{\omega_o} - 1\right)$$

Engineering Applications & Automotive Differential I

Automotive Differential Mechanism

An epicyclic bevel gear train located at the driven axle of motor vehicles.

- **Straight Path**: Left and right wheels rotate at identical speeds ($N_L = N_R = N_{\text{crown}}$).
- **Cornering**: Outer wheel travels a longer path; differential allows $N_L \neq N_R$ while maintaining constant average speed:

$$N_{\text{crown}} = \frac{N_L + N_R}{2}$$

Industrial Applications of Gear Systems

- **Planetary Gearboxes**: Used in automatic transmissions, wind turbines, aerospace actuators for maximum torque density in minimal volume.
- **Worm Drives**: Used in hoists and winches for self-locking high speed reduction ($> 50:1$).

Design & Selection Summary

Selection of gear drives depends on shaft alignment, speed reduction ratio, torque capacity, efficiency, noise/vibration limits, and manufacturing constraints.

Lecture Overview: Flywheels & Turning Moment Diagrams I

Course Information

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit 5.1**: Flywheels and Turning Moment Diagrams
- **Focus**: Energy fluctuation, turning moment diagrams, fluctuation coefficients, flywheel sizing, and rim stresses.

Lecture Overview: Flywheels & Turning Moment Diagrams II

Learning Objectives

1. Distinguish between the operating principles and functions of a **Flywheel** and a **Governor**.
2. Construct and analyze **Turning Moment Diagrams (TMD)** for single and multi-cylinder engines.
3. Calculate the **Maximum Fluctuation of Energy (ΔE)** using graphical area methods.
4. Evaluate the **Coefficients of Speed (C_s)** and **Energy (C_e)** fluctuation.
5. Size flywheels based on mass moment of inertia (I), rim velocity (v), and centrifugal stress (σ).
6. Apply flywheel energy storage principles to intermittent machinery such as **Punching Presses**.

Introduction to Flywheels & Comparison with Governors I

Definition of a Flywheel

A **flywheel** is an inertial energy storage device that absorbs energy during periods when power supply exceeds demand and releases energy when demand exceeds supply, maintaining speed variation within permissible limits during a single machine cycle.

Introduction to Flywheels & Comparison with Governors II

Comparison: Flywheel vs. Governor

Parameter	**Flywheel**	**Governor**
Primary Function	Controls cyclic speed variations due to engine torque fluctuations.	Controls mean speed variations due to external load changes.
Time Frame	Operates within a single revolution / cycle.	Operates over multiple revolutions / cycles.
Working Principle	Purely mechanical inertial storage ($E = \frac{1}{2} I \omega^2$).	Regulates working fluid / fuel supply via speed feedback mechanism.
Energy Action	Stores and releases kinetic energy passively.	Adjusts power output actively by regulating charge/fuel.

Concept of Turning Moment Diagrams (TMD) I

What is a Turning Moment Diagram?

A **Turning Moment Diagram** (also called Crank Effort Diagram) is a graphical representation of the turning moment (torque T) plotted on the ordinate against the crank angle (θ) on the abscissa for a complete operating cycle.

Concept of Turning Moment Diagrams (TMD) II

Key Mathematical Properties

- **Work Done per Cycle (W)**: Area under the T–θ curve.

$$W = \int_0^{\theta_{\text{cycle}}} T \, d\theta$$

- **Mean Resisting Torque (T_{mean})**: The uniform torque required to perform the same work over the cycle.

$$T_{\text{mean}} = \frac{W}{\theta_{\text{cycle}}} = \frac{\text{Work Done per Cycle}}{\text{Crank Angle per Cycle}}$$

- **Energy Balance**:
  - When $T > T_{\text{mean}}$: Engine accelerates; surplus energy is stored in the flywheel.
  - When $T < T_{\text{mean}}$: Engine decelerates; energy deficit is supplied by the flywheel.

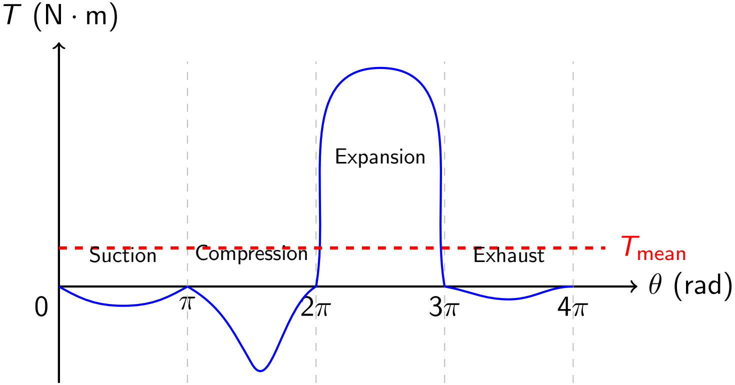
TMD for a Four-Stroke Single-Cylinder Engine I

Four-Stroke Engine Cycle Dynamics ($\theta = 0$ to 4π rad)

In a 4-stroke internal combustion engine, one complete cycle occurs over two revolutions ($720^\circ = 4\pi$ rad):

- **Suction Stroke** ($0 \to \pi$): Negative torque loop due to pressure drop below atmospheric.
- **Compression Stroke** ($\pi \to 2\pi$): Large negative loop as work is done on the gas mixture.
- **Expansion (Power) Stroke** ($2\pi \to 3\pi$): Massive positive loop delivering energy.
- **Exhaust Stroke** ($3\pi \to 4\pi$): Small negative loop to push exhaust gases out.

TMD for a Four-Stroke Single-Cylinder Engine II



Analytical Expression for Turning Moment I

Crank Effort Equation

The exact turning moment T acting on the crankshaft of a reciprocating engine is given by:

$$T = F_P \cdot r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$

where F_P is net piston effort, r is crank radius, l is connecting rod length, and $n = l/r$.
For large connecting rod ratios ($n \geq 4$), the expression simplifies to:

$$T \approx F_P \cdot r \left(\sin\theta + \frac{\sin 2\theta}{2n} \right)$$

Analytical Expression for Turning Moment II

Piston Effort Formulation

The net piston effort F_P is determined by accounting for gas pressure, reciprocating inertia, and friction:

$$F_P = F_g \pm F_I - F_f \pm W_R$$

- **Gas Force**: $F_g = p \cdot \frac{\pi}{4} D^2$
- **Reciprocating Inertia Force**: $F_I = m_R \cdot \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n}\right)$

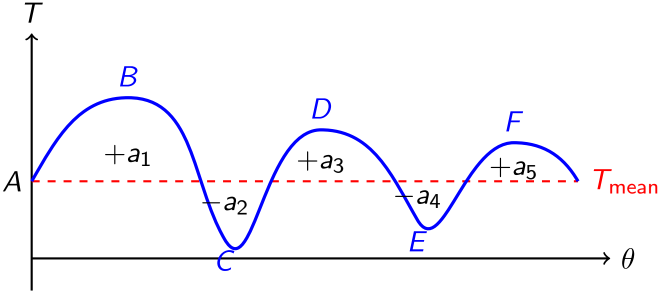
Determination of Maximum Fluctuation of Energy (ΔE) I

Energy Level Integration along T_{mean} Intersections

Let the energy of the flywheel at starting point A be $E_A = E$. The energy levels at subsequent intersection points B, C, D, E, F are calculated by algebraically accumulating loop areas a_1, a_2, a_3, a_4, a_5:

$$
\begin{aligned}
E_B &= E + a_1 \\
E_C &= E_B - a_2 = E + a_1 - a_2 \\
E_D &= E_C + a_3 = E + a_1 - a_2 + a_3 \\
E_E &= E_D - a_4 = E + a_1 - a_2 + a_3 - a_4 \\
E_F &= E_E + a_5 = E + a_1 - a_2 + a_3 - a_4 + a_5
\end{aligned}
$$

Determination of Maximum Fluctuation of Energy (ΔE) II



Maximum Fluctuation Definition

$$\Delta E = E_{\text{max}} - E_{\text{min}}$$

where E_{max} and E_{min} are the maximum and minimum energy values among all key intersection points.

Coefficients of Speed and Energy Fluctuation I

Coefficient of Fluctuation of Speed (C_s)

Defined as the ratio of maximum speed variation to mean speed:

$$C_s = \frac{N_1 - N_2}{N} = \frac{2(N_1 - N_2)}{N_1 + N_2} = \frac{\omega_1 - \omega_2}{\omega}$$

where N_1, N_2 are maximum and minimum speeds, and $N = \frac{N_1+N_2}{2}$ is mean speed.

Coefficient of Steadiness (m)

The reciprocal of the coefficient of fluctuation of speed:

$$m = \frac{1}{C_s} = \frac{N}{N_1 - N_2}$$

Coefficients of Speed and Energy Fluctuation II

Coefficient of Fluctuation of Energy (C_e)

Defined as the ratio of maximum energy fluctuation to work done per cycle:

$$C_e = \frac{\text{Maximum Fluctuation of Energy}}{\text{Work Done per Cycle}} = \frac{\Delta E}{W}$$

Flywheel Sizing and Inertia Calculations I

Derivation of Fluctuation Energy Equation

The kinetic energy stored in a flywheel rotating at angular velocity ω is $E = \frac{1}{2}I\omega^2$.

$$\Delta E = \frac{1}{2}I(\omega_1^2 - \omega_2^2) = \frac{1}{2}I(\omega_1 - \omega_2)(\omega_1 + \omega_2)$$
$$= I \cdot \omega \cdot (\omega_1 - \omega_2) = I \cdot \omega^2 \cdot \left(\frac{\omega_1 - \omega_2}{\omega}\right)$$

$$\Delta E = I\omega^2 C_s = mk^2\omega^2 C_s$$

Flywheel Sizing and Inertia Calculations II

Rim-Type Flywheel & Tensile Stress Limits

For a rim-type flywheel where rim radius $R \approx k$ and peripheral velocity $v = \omega R$:

$$\Delta E = mv^2 C_s$$

The centrifugal tensile stress σ developed in the rim is governed by material density ρ:

$$\sigma = \rho v^2 \implies v_{\text{max}} = \sqrt{\frac{\sigma_{\text{safe}}}{\rho}}$$

For gray cast iron ($\rho \approx 7200$ kg/m^3, $\sigma_{\text{safe}} \approx 7$ MPa), $v \leq 30$ m/s.

Flywheels in Punching Presses I

Working Cycle of a Punching Press

A punching press performs intermittent work: punching occurs during a small fraction t_1 of the total cycle time t.

- **Punching Energy Required (E_{punch})**:

$$E_{\text{punch}} = \text{Mean Punching Force} \times \text{Plate Thickness} = \frac{1}{2} F_{\text{max}} \cdot t_{\text{sheet}}$$

- **Continuous Motor Supply (E_{motor})**: Electric motor provides constant power P:

$$E_{\text{motor}} = P \cdot t$$

Flywheels in Punching Presses II

Energy Deficit Supplied by Flywheel

During the punching interval t_1, motor supplies energy $P \cdot t_1$, so the flywheel supplies the deficit:

$$\Delta E = E_{\text{punch}} - P \cdot t_1 = E_{\text{punch}} \left(1 - \frac{t_1}{t}\right)$$

This energy release slows down the flywheel from ω_1 to ω_2, which is re-accelerated during idling $(t - t_1)$.

Numerical Design Example: Flywheel Parameter Calculation

Problem Statement

A 4-stroke IC engine produces 50 kW power at 300 rpm. The coefficient of fluctuation of speed is $C_s = 0.02$. The maximum fluctuation of energy ΔE is 10% of the work done per cycle. Find the required mass moment of inertia I of the flywheel.

Step-by-Step Solution

1. **Mean Angular Velocity (ω)**:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi(300)}{60} = 31.416 \text{ rad/s}$$

2. **Work Done per Cycle (W)**: For a 4-stroke engine, cycles per second $= \frac{N}{2\times 60} = \frac{300}{120} = 2.5$ cycles/s.

$$W = \frac{\text{Power}}{\text{cycles/s}} = \frac{50000}{2.5} = 20,000 \text{ J/cycle}$$

3. **Maximum Energy Fluctuation (ΔE)**:

$$\Delta E = 0.10 \times 20,000 = 2,000 \text{ J}$$

4. **Required Inertia (I)**:

$$I = \frac{\Delta E}{\omega^2 C_s} = \frac{2000}{(31.416)^2 \times 0.02} \approx 101.32 \text{ kg} \cdot \text{m}^2$$

Lecture Summary & Key Design Equations I

Core Equations Summary

Parameter / Concept	**Governing Expression**
Kinetic Energy Stored	$E = \frac{1}{2} I\omega^2 = \frac{1}{2} mk^2\omega^2$
Max Fluctuation of Energy	$\Delta E = I\omega^2 C_s = mk^2\omega^2 C_s = mv^2 C_s$
Coeff. of Speed Fluctuation	$C_s = \frac{N_1 - N_2}{N} = \frac{\omega_1 - \omega_2}{\omega}$
Coeff. of Energy Fluctuation	$C_e = \frac{\Delta E}{W}$
Centrifugal Rim Stress	$\sigma = \rho v^2 = \rho\omega^2 R^2$
Punching Press Energy Deficit	$\Delta E = E_{\text{punch}} \left(1 - \frac{t_1}{t}\right)$

Lecture Summary & Key Design Equations II

Key Engineering Takeaways

- Flywheels handle cyclic kinetic energy variations; governors regulate supply based on external load.
- Maximum fluctuation of energy (ΔE) determines flywheel mass moment of inertia (I).
- Maximum allowable peripheral velocity v is limited by material centrifugal stress ($\sigma = \rho v^2$).

Lecture Overview: Flywheels & Turning Moment Diagrams I

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Topic:** 5.1 Flywheels and Turning Moment Diagrams
- **Unit:** Flywheels and Governors

Key Learning Objectives

- Understand the mechanical engineering principles of energy storage in rotating systems.
- Analyze cyclic torque fluctuations in reciprocating engines and industrial machinery.
- Formulate principles of flywheel sizing, speed regulation, and structural safety limits.

Introduction to Flywheels I

Definition & Primary Purpose

A **flywheel** is a heavy rotating mass that acts as an energy reservoir, mitigating cyclic speed fluctuations caused by variations in engine torque or load torque during a single operating cycle.

Introduction to Flywheels II

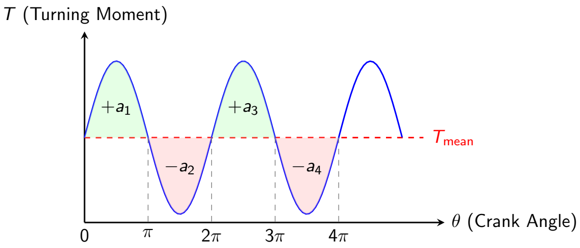
Fundamental Working Principles

- **Kinetic Energy Storage:** The flywheel absorbs energy when torque supply exceeds demand (accelerating) and releases energy when demand exceeds supply (decelerating).
- **Cyclic Speed Regulation:** Regulates speed variations *within a single cycle* (short-term fluctuations), unlike governors which regulate mean speed across multiple cycles under varying load conditions.
- **Engineering Benefits:** Reduces engine vibration, allows smaller driving motors, and protects driven machinery from severe shock loads.

Turning Moment Diagram (TMD)

Definition of TMD

A **Turning Moment Diagram** (or Crank Effort Diagram) plots the turning moment (T) exerted on the crankshaft as a function of the crank angle (θ).



Key Mathematical Concepts

- **Work Done per Cycle:** Area under curve $W = \int_0^{\theta_{\text{cycle}}} T\, d\theta = T_{\text{mean}} \times \theta_{\text{cycle}}$.
- **Excess/Deficit Energy Areas:** Areas above T_{mean} $(+a_1, +a_3)$ store excess energy in the flywheel; areas below $(-a_2, -a_4)$ extract energy.

TMD: Single vs. Multi-Cylinder Engines I

Engine Configuration Impact

Turning moment diagrams vary significantly based on engine cylinder count and stroke timing, directly determining flywheel inertia requirements.

Parameter	**Single-Cylinder 4-Stroke**	**Multi-Cylinder Engine**
Torque Profile	Single high peak during expansion stroke (180°–360°); negative work during other 3 strokes.	Overlapping power strokes phased uniformly (e.g., 120° for 3-cylinder).
Energy Fluctuation (ΔE)	Very large peak-to-valley energy difference.	Small energy fluctuations as peaks fill in valleys.
Flywheel Requirement	Heavy, large-diameter flywheel needed for smoothing.	Substantially lighter flywheel required for equivalent speed regulation.

Fluctuation of Energy and Speed

Quantitative Metrics

- **Maximum Fluctuation of Energy (ΔE):** The difference between maximum energy (E_{max}) and minimum energy (E_{min}) stored by the flywheel during a complete cycle:

$$\Delta E = E_{\text{max}} - E_{\text{min}}$$

- **Coefficient of Fluctuation of Speed (C_s):** Ratio of maximum speed variation to mean angular speed:

$$C_s = \frac{\omega_1 - \omega_2}{\omega} = \frac{N_1 - N_2}{N}$$

  where $\omega = \frac{\omega_1 + \omega_2}{2}$ is the mean angular speed.
- **Coefficient of Fluctuation of Energy (C_e):**

$$C_e = \frac{\Delta E}{\text{Work done per cycle}}$$

Typical Permissible C_s Values

- **Electrical Alternator Drives:** $C_s \approx 0.003 - 0.005$ ($0.3\% - 0.5\%$ to avoid grid synchronization phase drift).
- **Pumping Machinery:** $C_s \approx 0.03 - 0.05$ ($3\% - 5\%$).
- **Punching Presses & Machine Tools:** $C_s \approx 0.10 - 0.20$ ($10\% - 20\%$).

Cumulative Energy Analysis I

Determination of Maximum Energy Fluctuation

To determine ΔE, evaluate the cumulative energy at every intersection of the turning moment curve with the mean torque line T_{mean}.

Cumulative Energy Analysis II

Step-by-Step Cumulative Method

1. Let initial flywheel energy at point A be E.
2. Energy at subsequent points B, C, D, E, F:

$$\begin{aligned} E_B &= E + a_1 \\ E_C &= E + a_1 - a_2 \\ E_D &= E + a_1 - a_2 + a_3 \\ E_E &= E + a_1 - a_2 + a_3 - a_4 \\ E_F &= E \quad \text{(Cycle completes and closes)} \end{aligned}$$

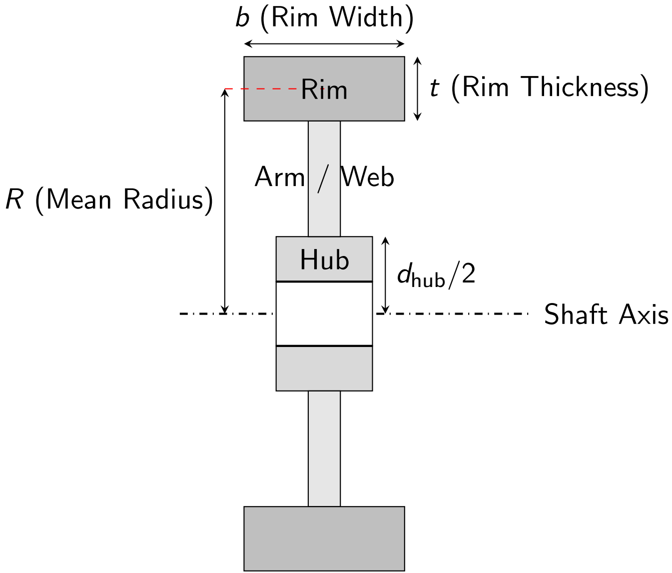
3. Identify $E_{\text{max}} = \max(E_A, E_B, E_C, \dots)$ and $E_{\text{min}} = \min(E_A, E_B, E_C, \dots)$.
4. Compute $\Delta E = E_{\text{max}} - E_{\text{min}}$.

Flywheel Rim Geometry and Construction I

Structural Components of Rim Flywheels

Rim-type flywheels concentrate mass in the outer rim to maximize the radius of gyration ($k \approx R$).

Flywheel Rim Geometry and Construction II



Equation of Flywheel Motion I

Derivation of Flywheel Energy Formula

- Kinetic energy of rotation: $E = \frac{1}{2}I\omega^2$
- Kinetic energy fluctuation:

$$\Delta E = E_1 - E_2 = \frac{1}{2}I(\omega_1^2 - \omega_2^2) = \frac{1}{2}I(\omega_1 - \omega_2)(\omega_1 + \omega_2)$$

- Substituting mean speed $\omega = \frac{\omega_1+\omega_2}{2}$ and speed fluctuation $C_s = \frac{\omega_1-\omega_2}{\omega}$:

$$\Delta E = I\omega^2 C_s$$

- In terms of rim mass m and radius of gyration k (where $I = mk^2$):

$$\Delta E = mk^2\omega^2 C_s = mv^2 C_s$$

where $v = \omega R$ is the mean peripheral rim velocity.

Flywheels in Punching Press Applications I

Punching Press Mechanics

Punching presses use flywheels to execute high-force shearing operations using low-power continuous motors.

Flywheels in Punching Press Applications II

Energy Balance Calculations

- **Punching Energy Required:** Energy required to punch a hole of thickness t with maximum force F_{max}:

$$E_{\text{punch}} = \frac{1}{2} F_{\text{max}}\, t$$

- **Motor Energy Supplied:** During total cycle time t_{cycle}, motor supplies:

$$E_{\text{motor}} = P_{\text{motor}} \times t_{\text{cycle}}$$

- **Energy Supplied by Flywheel:** During the brief punching duration t_{punch}, the flywheel provides energy:

$$\Delta E = E_{\text{punch}} - E_{\text{motor}} \left(\frac{t_{\text{punch}}}{t_{\text{cycle}}} \right)$$

Comparative Summary: Flywheels vs. Governors I

Feature	Flywheel	Governor
Function	Regulates cyclic speed fluctuations within a single cycle.	Regulates mean speed across multiple cycles under varying load.
Operation	Passive energy storage (absorbs/releases kinetic energy).	Active control (adjusts fuel/working fluid flow rate).
Sensitivity	Insensitive to mean engine load changes.	Sensitive to load variations.
Execution	Continuous throughout the cycle.	Intermittent (activates only when mean speed shifts).

Design Considerations & Safety Limits I

Centrifugal Hoop Stress Limit

Centrifugal force induces tensile hoop stress (σ_t) in the rotating flywheel rim:

$$\sigma_t = \rho v^2$$

where ρ is material density and v is linear rim speed ($v = \omega R$).

Material & Speed Thresholds

- **Cast Iron Rim Limit:** Allowable tensile stress limits $v \leq 30$ m/s to avoid catastrophic bursting failure.
- **High-Speed Applications:** Ductile iron, cast steel, or composite materials allow velocities up to 100 m/s or higher.
- **Optimization Trade-off:** Sizing balances kinetic capacity ($\Delta E \propto mR^2$), spatial constraints, and safe structural stress limits.

Lecture 37: Overview of Governors I

Course Information

- **Course**: Theory of Machines and Mechanisms (DI03000141)
- **Unit**: Flywheel and Governors
- **Lecture**: 37 – 5.2 Governors

Lecture 37: Overview of Governors II

Lecture Objectives

- Understand the fundamental principle and function of mechanical governors.
- Classify governors into Centrifugal (Pendulum, Dead Weight, Spring-Controlled) and Inertia types.
- Analyze the working, schematic layout, and governing equations of Watt, Porter, Proell, and Hartnell governors.
- Evaluate governor stability, controlling force curves, and key performance characteristics (sensitiveness, isochronism, hunting, effort, and power).
- Compare the operational mechanisms of flywheels and governors.

Introduction to Governors I

Definition & Primary Function

A **governor** is a mechanical speed-control device designed to automatically regulate the supply of working fluid (fuel, steam, gas) to an engine to maintain its mean speed within prescribed limits despite variations in external load.

Principle of Operation

- **Load Increase**: Engine speed drops $\rightarrow$ Governor mechanism increases working fluid throttle opening $\rightarrow$ Mean speed is restored.
- **Load Decrease**: Engine speed rises $\rightarrow$ Governor mechanism decreases working fluid throttle opening $\rightarrow$ Mean speed is restored.

Introduction to Governors II

Fundamental Distinction: Flywheel vs. Governor

- **Flywheel**: Reduces *cyclic* speed fluctuations caused by internal turning moment variations within a **single thermodynamic cycle**.
- **Governor**: Controls *long-term* speed variations caused by changes in **external load** over multiple cycles.

Classification of Governors

Broad Categories

Governors are broadly classified based on their operating forces:

1. Centrifugal Governors

Operate by balancing the centrifugal force of rotating balls against a controlling force.

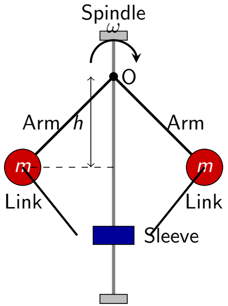
- **Pendulum Type**:
  - Watt Governor
- **Loaded Type**:
  - *Dead-Weight*: Porter, Proell
  - *Spring-Controlled*: Hartnell, Hartung, Wilson-Hartnell

2. Inertia Governors

Operate on both angular acceleration and centrifugal force of rotating masses.

- Governor responds to rate of change of speed ($\frac{d\omega}{dt}$) as well as speed (ω).
- Offers significantly faster response time than purely centrifugal governors.
- Less commonly used due to practical balancing challenges.

Watt Governor (Pendulum Type)



Working & Governing Equation

Invented by James Watt. Uses two flyballs of mass m attached to arms pivoted to a central spindle.

- Height h: Vertical distance from ball center to upper arm pivot intersection.
- Balancing moments about pivot O:

$$F_c \cdot h = m \cdot g \cdot r \implies m\omega^2 r \cdot h = mgr$$

$$h = \frac{g}{\omega^2} = \frac{895}{N^2} \text{ meters}$$

Key Limitation

At speeds above 80 rpm, h becomes extremely small (e.g., at $N = 300$ rpm, $h \approx 9.9$ mm), rendering the governor insensitive to speed variations.

Porter Governor (Dead Weight Type) I

Operating Principle

The Porter governor enhances sensitivity at higher speeds by adding a heavy dead weight (mass M) on the sleeve, providing additional downward restoring force.

Governing Equation (Without Friction)

Taking moments about the instantaneous center or using force equilibrium:

$$N^2 = \left[\frac{m + \frac{M}{2}(1+k)}{m}\right] \cdot \frac{895}{h}$$

where $k = \frac{\tan\beta}{\tan\alpha}$, α is upper arm angle with vertical, and β is lower link angle with vertical.

- **Equal Arm & Link Lengths** ($\alpha = \beta \implies k = 1$):

$$N^2 = \left(\frac{m + M}{m}\right) \cdot \frac{895}{h}$$

Porter Governor (Dead Weight Type) III

Frictional Effect (f at Sleeve)

Sleeve friction force f opposes sleeve motion (downward when speed increases, upward when speed drops):

$$N^2 = \left[\frac{m \cdot g + \frac{M \cdot g \pm f}{2}(1 + k)}{m \cdot g} \right] \cdot \frac{895}{h}$$

Proell Governor I

Design Improvement over Porter Governor

In a Proell governor, the flyballs are mounted on upward extensions of the lower links rather than directly at the arm-link junction.

Key Operational Advantages

- Extension arms change the moment arm ratio, resulting in greater sleeve lift for a given change in speed.
- Requires a significantly smaller flyball mass m compared to a Porter governor for the same sleeve lift and controlling force.

Proell Governor II

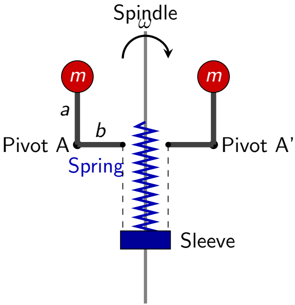
Governing Equation

$$N^2 = \frac{a}{e} \cdot \left[\frac{m + \frac{M}{2}(1+k)}{m} \right] \cdot \frac{895}{h}$$

where:

- a: Vertical distance from upper arm pivot to the center of the ball.
- e: Length of the lower link.
- $k = \frac{\tan\beta}{\tan\alpha}$.

Hartnell Governor (Spring-Controlled Type)



Working Principle

Utilizes a central helical spring pressing against the sleeve and bell-crank levers carrying flyballs.

- Vertical arm (length a): Holds flyball of mass m.
- Horizontal arm (length b): Drives the sleeve.

Spring Stiffness Formula (s)

Taking equilibrium moments at minimum radius r_1 and maximum radius r_2:

$$s = 2\left(\frac{F_2 - F_1}{r_2 - r_1}\right) \cdot \left(\frac{a}{b}\right)^2$$

where F_1, F_2 are centrifugal forces at r_1, r_2.

Design Advantages

Extremely compact, operates at high rotational speeds, highly adaptable by altering spring pre-compression.

Controlling Force and Stability I

Controlling Force (F_c)

The resultant inward radial force acting on the flyballs to balance centrifugal force at any equilibrium speed:

$$F_c = m \cdot \omega^2 \cdot r$$

Stability Criterion

A governor is **stable** if the equilibrium speed increases as the radius of rotation increases.

$$\frac{dF_c}{dr} > \frac{F_c}{r}$$

Controlling Force and Stability II

Controlling Force Curves ($F_c = a \cdot r + b$)

- **Stable Governor**: $b > 0$ (positive intercept on F_c axis).
- **Unstable Governor**: $b < 0$ (negative intercept on F_c axis).
- **Isochronous Governor**: $b = 0$ ($F_c = a \cdot r$, line passes through the origin).

Governor Performance Characteristics I

1. Sensitiveness

Ratio of speed range to mean speed:

$$\text{Sensitiveness} = \frac{N_2 - N_1}{N} = \frac{2(N_2 - N_1)}{N_1 + N_2}$$

A sensitive governor produces significant sleeve lift for small speed variations.

2. Isochronism

A governor is **isochronous** when the equilibrium speed is constant for all radii of rotation $(N_1 = N_2 = N)$. Sensitiveness is theoretically infinite.

Governor Performance Characteristics II

3. Hunting

Continuous, violent speed fluctuations above and below the mean speed. Caused by an oversensitive governor that over-corrects fuel supply repeatedly.

4. Effort and Power

- **Governor Effort**: Mean force exerted on the sleeve for a given percentage change in speed.
- **Governor Power**: Work done on sleeve during lift (Power $=$ Effort $\times$ Sleeve Lift).

Comparison: Flywheel vs. Governor I

Table: Comparative Analysis of Flywheels and Governors

Parameter & **Flywheel** & **Governor**
Primary Goal & Controls cyclic speed fluctuations & Controls mean speed over time
Time Frame & Within a single cycle & Over multiple cycles
Source of Variation & Internal engine torque changes & External load changes
Working Mechanism & Stores and releases kinetic energy & Regulates working fluid quantity
Control Action & Continuous & automatic & Intermittent (load-dependent)
Necessity & Optional if torque variation is small & Mandatory for speed stability

Summary of Governing Equations I

Table: Summary of Governor Types and Key Formulas

Governor Type & **Category** & **Governing Equation / Property**
Watt & Pendulum & $h = \frac{895}{N^2}$ (sensitive only at $N < 80$ rpm)
Porter & Dead-Weight & $N^2 = \left[\frac{m+\frac{M}{2}(1+k)}{m}\right] \frac{895}{h}$
Proell & Dead-Weight & $N^2 = \frac{a}{e}\left[\frac{m+\frac{M}{2}(1+k)}{m}\right] \frac{895}{h}$
Hartnell & Spring-Loaded & $s = 2\left(\frac{F_2 - F_1}{r_2 - r_1}\right)\left(\frac{a}{b}\right)^2$
Stability & Performance & $\frac{dF_c}{dr} > \frac{F_c}{r}$ $(F_c = a \cdot r + b,\ b > 0)$

Course & Unit Information

Course: Theory of Machines and Mechanisms (GTU DI03000141)
Unit 5: Flywheel and Governors
Lecture 38: 5.2 Governors

Lecture 38: 5.2 Governors – Overview II

Learning Objectives

- Understand the fundamental function of a governor as an automatic speed control feedback mechanism.
- Classify governors into Centrifugal vs. Inertia types, and Gravity-Controlled vs. Spring-Controlled types.
- Derive and analyze governing speed equations for Watt, Porter, Proell, and Hartnell governors.
- Define and evaluate performance parameters: Sensitiveness, Stability, Isochronism, Hunting, Effort, and Power.
- Distinguish between the operational roles of Flywheels and Governors in prime movers.

Definition & Primary Function

A **governor** is an automatic speed control device designed to regulate the mean speed of an engine or prime mover when variations in external load occur.

Introduction to Governors II

Feedback Mechanism

- Operates as a closed-loop feedback controller where:
  - **Input:** Engine speed variation / error ($\Delta N = N - N_{\text{mean}}$).
  - **Action:** Mechanical movement of governor flyballs and sleeve.
  - **Output:** Adjustment of throttle valve regulating fuel or working fluid supply.
- **Load Increase** $\rightarrow$ Engine speed decreases $\rightarrow$ Governor increases fuel supply.
- **Load Decrease** $\rightarrow$ Engine speed increases $\rightarrow$ Governor throttles back fuel supply.

Applications

Essential in steam turbines, internal combustion (IC) engines, hydroelectric turbines, and gas turbines operating under varying electrical or mechanical loads.

Classification of Governors I

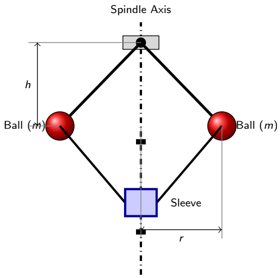
Primary Classification Categories

1. **Centrifugal Governors:** Utilize centrifugal force of rotating flyballs balanced against gravity dead-weights or spring forces.
2. **Inertia Governors:** Utilize rotational inertia and angular acceleration ($\alpha = d\omega/dt$) for rapid dynamic response.

Taxonomy of Centrifugal Governors

Category	**Governor Types**	**Control Mechanism**
Gravity-Controlled	Watt Governor	Simple arm pivot (no dead-weight)
Gravity-Controlled	Porter Governor	Heavy dead-weight M on sleeve
Gravity-Controlled	Proell Governor	Dead-weight M + balls on extension links
Spring-Controlled	Hartnell Governor	Helical compression spring on spindle
Spring-Controlled	Hartung Governor	Horizontal springs acting directly on balls
Spring-Controlled	Wilson-Hartnell	Main central spring + auxiliary arm springs

Centrifugal Governors: The Watt Governor



Governing Speed Equation

Taking moments about the upper pivot point:

$$F_c \cdot h = mg \cdot r \implies (m\omega^2 r)h = mgr$$

$$h = \frac{g}{\omega^2} = \frac{9.81}{\left(\frac{2\pi N}{60}\right)^2} = \frac{895}{N^2} \text{ meters}$$

where N is speed in rpm and h is governor height in meters.

Speed Limitation of Watt Governor

- At $N = 60$ rpm $\implies$ $h = 0.248$ m $= 248$ mm.
- At $N = 80$ rpm $\implies$ $h = 0.140$ m $= 140$ mm.
- At $N > 80$ rpm, the rate of change $\frac{dh}{dN} = -\frac{1790}{N^3}$ becomes extremely small. The sleeve lift is too minuscule to actuate the throttle valve, rendering the Watt governor insensitive at high speeds.

Porter Governor (Dead-Weight Loaded)

Construction & Design Principle

- A heavy central dead-weight of mass M (weight Mg) is mounted on the central sleeve.
- Increases the downward restoring force on the sleeve without increasing the inertia mass m of the flyballs.
- Enables operation at higher speeds with high sensitivity.

Governing Equations

Let $\theta =$ angle of upper arm to spindle, $\beta =$ angle of lower arm to spindle, and $q = \frac{\tan\beta}{\tan\theta}$.

- **General Governing Speed Formula (without friction):**

$$N^2 = \frac{895}{h} \cdot \left[\frac{m + \frac{M}{2}(1+q)}{m}\right]$$

- **Equal Arms Pivoted on Spindle Axis ($q = 1$):**

$$N^2 = \frac{895}{h} \cdot \left(\frac{m + M}{m}\right)$$

- **Inclusion of Sleeve Frictional Resistance (F):**

$$N^2 = \frac{895}{h} \cdot \left[\frac{mg + \frac{Mg \pm F}{2}(1+q)}{mg}\right]$$

  where $+F$ applies when sleeve moves UP (speed rising) and $-F$ applies when sleeve moves DOWN (speed falling).

Proell Governor I

Design Feature

- Modifications to Porter governor: Flyballs are attached to rigid extension links extending outwards from the lower arms.
- Ball pivot center is positioned at distance FM from link line BM.
- Provides greater sleeve lift for a given speed change compared to the Porter governor.

Governing Speed Formula

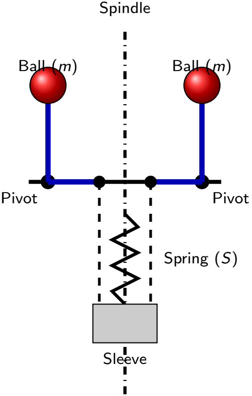
For equal arm lengths ($q = 1$) with flyball mounted at extension distance FM and lower arm length BM:

$$N^2 = \frac{895}{h} \cdot \frac{FM}{BM} \cdot \left(\frac{m + M}{m} \right)$$

Proell Governor II

Comparison of Dead-Weight Governors

Governor Type	**Governing Equation (N^2)**	**Key Advantage**
Watt	$\frac{895}{h}$	Simple, low speed (<80 rpm)
Porter	$\frac{895}{h}\left(\frac{m+M}{m}\right)$	Operates at higher speeds ($100-300$ rpm)
Proell	$\frac{895}{h}\frac{FM}{BM}\left(\frac{m+M}{m}\right)$	Higher sensitivity, smaller ball mass m



Construction

Uses bell crank levers carrying flyballs on vertical arms of length a and sleeve rollers on horizontal arms of length b. A central helical spring applies force S.

Spring-Controlled Governors: Hartnell Governor II

Governing Equations & Spring Stiffness

- At minimum speed N_1 (radius r_1): $S_1 = 2F_{c1}\left(\frac{a}{b}\right)$
- At maximum speed N_2 (radius r_2): $S_2 = 2F_{c2}\left(\frac{a}{b}\right)$
- Sleeve lift $h_s = (r_2 - r_1)\frac{b}{a}$
- Spring Stiffness (s):

$$s = \frac{S_2 - S_1}{h_s} = 2\left(\frac{a}{b}\right)^2 \left(\frac{F_{c2} - F_{c1}}{r_2 - r_1}\right)$$

Governor Performance Characteristics I

1. Sensitiveness (S_n)

The range of speed over which the governor operates relative to its mean speed:

$$S_n = \frac{N_2 - N_1}{N_{\text{mean}}} = \frac{2(N_2 - N_1)}{N_1 + N_2}$$

A governor with a smaller speed range for a given sleeve lift is more sensitive.

2. Stability

A governor is **stable** if there is only one unique equilibrium radius of rotation r for each speed N within the working range. Speed N must increase monotonically as radius r increases ($\frac{dN}{dr} > 0$).

Governor Performance Characteristics II

3. Isochronism

- Theoretical limit where speed range is zero ($N_1 = N_2 = N_{\text{mean}}$).
- Sensitiveness becomes infinite ($S_n = 0$ denominator).
- Sleeve moves through full stroke for an infinitesimal speed variation.

4. Hunting

Continuous, violent oscillations of sleeve and speed around mean position due to over-sensitivity or friction lag in valve linkages.

Governor Effort and Power I

Governor Effort (P)

The mean force exerted by the governor at the sleeve during a specified fractional speed variation $c = \frac{\Delta N}{N}$:

- **Porter Governor Effort (Zero Friction):**

$$P = c \cdot (m + M)g$$

- **Porter Governor Effort with Friction F:**

$$P = c \cdot (mg + Mg + F)$$

Governor Effort and Power II

Governor Power (E)

The work done at the sleeve during the movement corresponding to a given speed change:

$$\text{Power } E = \text{Mean Effort } P \times \text{Sleeve Lift } x$$

- For Porter governor with equal arms ($x \approx 2h\frac{2c}{1+2c}$):

$$E \approx 2c^2(m+M)gh$$

Engineering Significance

High effort and power are essential to overcome mechanical resistance of throttle valve linkages without suffering speed lag.

Comparison: Flywheel vs. Governor I

Parameter	Flywheel	Governor
Primary Objective	Regulates cyclic speed fluctuations within a single cycle.	Regulates mean engine speed across multiple cycles under variable load.
Control Mechanism	Passive kinetic energy storage ($E_k = \frac{1}{2}I\omega^2$).	Active feedback regulation of fuel/working fluid rate.
Cause of Speed Variation	Fluctuations in internal engine turning moment (torque).	Changes in external load demand on prime mover.
Control Loop	Open-loop physical inertia.	Closed-loop speed feedback system.
Action Duration	Intermittent, intra-cycle (fractions of a second).	Continuous, multi-cycle long term regulation.
Essentiality	Essential when engine torque output is non-uniform.	Essential when prime mover power demand fluctuates.

Table: Comparative Summary: Flywheel vs. Governor

Summary of Formulas & GTU Exam Review I

Summary of Key Governing Equations

Governor Type	**Governing Formula**
Watt	$h = \frac{895}{N^2}$
Porter (Equal arms)	$N^2 = \frac{895}{h}\left(\frac{m+M}{m}\right)$
Porter with Friction	$N^2 = \frac{895}{h}\left[\frac{mg+\frac{Mg\pm F}{2}(1+q)}{mg}\right]$
Proell (Equal arms)	$N^2 = \frac{895}{h}\frac{FM}{BM}\left(\frac{m+M}{m}\right)$
Hartnell Stiffness	$s = 2\left(\frac{a}{b}\right)^2\left(\frac{F_{c2}-F_{c1}}{r_2-r_1}\right)$
Sensitiveness	$S_n = \frac{2(N_2-N_1)}{N_1+N_2}$

Summary of Formulas & GTU Exam Review II

GTU Exam Review Questions

1. Explain why a Watt governor is ineffective at speeds above 80 rpm.
2. Derive an expression for the height of a Porter governor taking friction into account.
3. Define Isochronism, Stability, and Hunting in centrifugal governors.
4. Compare the function and operating principle of a flywheel and a governor.

Lecture Overview: Comparison between Flywheel and Governor I

- **Course Context:** Theory of Machines and Mechanisms (GTU Course DI03000141).
- **Core Topic:** Detailed comparative analysis of energy regulators and speed controllers in prime movers.
- **Primary Distinction:**
  - **Flywheel:** Functions as an energy reservoir regulating cyclic speed fluctuations caused by internal torque variation during each operating cycle.
  - **Governor:** Functions as a speed controller regulating mean speed variations caused by external load variations over longer periods.
- **Key Learning Objectives:**
  - Understand fundamental thermodynamic and kinematic principles governing flywheels and governors.

Lecture Overview: Comparison between Flywheel and Governor II

- Analyze mathematical models for energy fluctuation (ΔE) and governor performance (sensitivity, stability, isochronism).
- Differentiate between open-loop inertia control and closed-loop feedback control.
- Evaluate their cooperative integration in multi-cylinder engines and powerplants.

Fundamental Definitions and Operating Principles I

Flywheel Operating Principle

A heavy rotating mass that stores excess mechanical energy as kinetic energy during periods of excess power supply and releases it during periods of energy deficit within a single thermodynamic cycle.

- Operates purely on physical inertia ($E_k = \frac{1}{2} I\omega^2$).
- Reduces cyclic fluctuations of speed ($\Delta\theta$ within a single cycle).
- Passive mechanism: Does not control or alter the fuel/working fluid input to the engine.

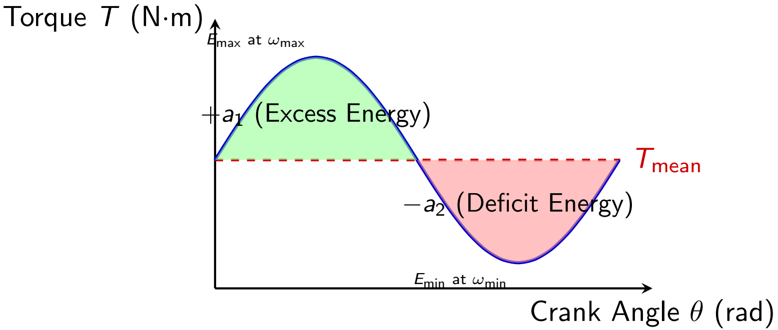
Fundamental Definitions and Operating Principles II

Governor Operating Principle

A speed regulation device that automatically adjusts the supply of working fluid (fuel/steam) to match the engine output power with varying external load demands.

- Operates on dynamic force equilibrium (centrifugal force balanced against sleeve weight or spring force).
- Controls mean engine speed (N_{mean}) over multiple revolutions and operational shifts.
- Active mechanism: Uses closed-loop feedback control to manipulate throttle position.

Turning Moment Diagram and Flywheel Energy Storage I



- **Excess Energy Area ($+a_1$):** When $T_{engine} > T_{mean}$, excess work is converted into kinetic energy, accelerating the flywheel rotor from ω_{min} to ω_{max}.

Turning Moment Diagram and Flywheel Energy Storage II

- **Deficit Energy Area ($-a_2$):** When $T_{\text{engine}} < T_{\text{mean}}$, the flywheel releases kinetic energy to drive the load, decelerating from ω_{max} to ω_{min}.
- **Maximum Fluctuation of Energy (ΔE):** $\Delta E = E_{\text{max}} - E_{\text{min}} = a_1 = a_2$ (at steady state).

The Flywheel: Mathematical Modeling

- **Dynamic Equation of Motion:**

$$T_{\text{engine}}(\theta) - T_{\text{load}} = I\alpha = I\frac{d\omega}{dt}$$

where $I = mk^2$ is the mass moment of inertia, m is the rim mass, and k is the radius of gyration.

- **Coefficient of Fluctuation of Speed (C_s):**

$$C_s = \frac{\omega_1 - \omega_2}{\omega_{\text{mean}}} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

Typical values of C_s:

  - Electrical Alternators: $C_s \approx 0.005$ (0.5%)
  - Internal Combustion Engines: $C_s \approx 0.01 - 0.02$ ($1\% - 2\%$)
  - Punching Presses & Shears: $C_s \approx 0.10 - 0.20$ ($10\% - 20\%$)

- **Maximum Fluctuation of Energy (ΔE):**

$$\Delta E = \frac{1}{2}I(\omega_1^2 - \omega_2^2) = \frac{1}{2}I(\omega_1 - \omega_2)(\omega_1 + \omega_2) \approx I\omega_{\text{mean}}^2 C_s = mk^2\omega_{\text{mean}}^2 C_s$$

- **Key Insight:** Flywheels limit the *amplitude* of cyclic speed oscillations, but cannot adjust the *mean speed* over changing load demands.

The Governor: Mathematical Modeling I

- **Primary Role:** Adjusts working fluid control valve to maintain mean equilibrium speed under varying external resistance.
- **Sensitiveness:**

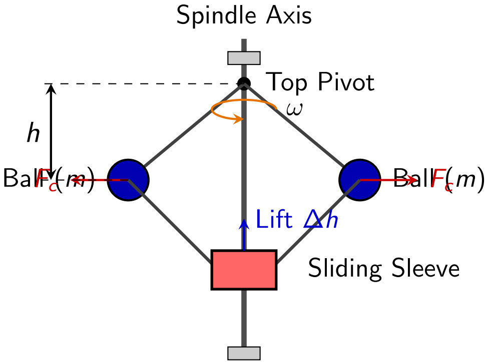
$$\text{Sensitiveness} = \frac{N_1 - N_2}{N_{\text{mean}}} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

  High sensitivity implies a large sleeve movement for a tiny change in speed.
- **Governor Stability:** A governor is stable if there is a unique equilibrium radius r of the revolving balls for every speed N within its operating range.
- **Isochronism:** An extreme case of sensitivity where the equilibrium speed remains constant across all radii of rotation $(N_1 = N_2)$. The sensitiveness becomes infinite.

The Governor: Mathematical Modeling II

- **Hunting:** Violent, continuous fluctuation of engine speed around the mean speed caused by an over-sensitive governor reacting with time lag.
- **Governor Effort and Power:**
  - *Effort:* Mean force exerted on the sleeve during a given percentage change in speed.
  - *Power:* Work done by the sleeve during the movement corresponding to that speed change (Power $=$ Effort $\times$ Sleeve Lift).

Centrifugal Governor Skeleton Diagram I



- Rotational speed ω produces centrifugal force $F_c = m\omega^2 r$ on governor balls.

Centrifugal Governor Skeleton Diagram II

- Equilibrium for simple Watt Governor (taking moments about top pivot):

$$mgr = F_c h = (m\omega^2 r)h \implies h = \frac{g}{\omega^2} = \frac{895}{N^2} \text{ meters}$$

- When $\omega \uparrow$, $F_c \uparrow$, sleeve moves upward (Δh), closing throttle valve to reduce fuel supply.

- **Nature of Regulation:**
  - *Flywheel:* Manages internal, short-term torque imbalance ($T_{\text{engine}} - T_{\text{mean}}$) during every revolution.
  - *Governor:* Manages external, long-term power imbalance ($P_{\text{generation}} - P_{\text{demand}}$) over extended operating periods.
- **Time Scale of Operation:**
  - *Flywheel:* Micro-seconds to milliseconds (within a fraction of a single engine stroke).
  - *Governor:* Seconds to minutes (during dynamic load switching or power grid fluctuations).
- **Control Loop Architecture:**
  - *Flywheel:* Open-loop, passive dynamic element. Directly absorbs kinetic energy without feedback components.
  - *Governor:* Closed-loop active control system comprising sensor (flyballs), controller (linkage mechanism), and actuator (throttle valve).

Key Physical and Performance Differences I

- **Mass and Inertia Requirements:**
  - *Flywheel:* Very heavy mass with high moment of inertia ($I = mk^2$). Designed specifically to store massive amounts of kinetic energy.
  - *Governor:* Lightweight assembly with minimal inertia to ensure rapid dynamic response without control lag.
- **Energy Storage Capability:**
  - *Flywheel:* Stores up to megajoules (MJ) of mechanical energy ($E = \frac{1}{2}I\omega^2$).
  - *Governor:* Stores negligible kinetic energy; acts strictly as an information and control mechanism.
- **Operational Continuity:**
  - *Flywheel:* Operates continuously across every degree of crankshaft rotation.
  - *Governor:* Operates intermittently, remaining stationary relative to the sleeve except during load variations.

Cooperative Actions in Engines and Powerplants I

Synergistic Co-existence

In practical prime movers (e.g., IC engines driving electrical generators, steam turbines, rolling mills), flywheels and governors are not substitutes for one another; they perform distinct complementary roles.

- **Transient Stabilization:** When a heavy load is suddenly shed, the governor requires time to throttle fuel due to mechanical latency. The flywheel absorbs the initial surge of kinetic energy, preventing sudden catastrophic overspeeding.
- **Elimination of Governor Hunting:** Without a flywheel, rapid cyclic torque fluctuations would cause the shaft speed to oscillate wildly during each stroke. This speed noise would trigger governor ball oscillations, leading to severe hunting.
- **System Integrity:**

Cooperative Actions in Engines and Powerplants II

- Engine with Governor only $\rightarrow$ Severe cyclic vibration and potential stall during high torque demand strokes.
- Engine with Flywheel only $\rightarrow$ Engine runs away under zero load or stalls under sustained high load.

Summary Comparison Matrix I

Parameter	**Flywheel**	**Governor**
Primary Function	Regulates cyclic speed variation within each operating cycle.	Regulates mean engine speed over varying load conditions.
Control Cause	Internal torque variation ($T_{\text{engine}} \neq T_{\text{mean}}$).	External load change ($T_{\text{mean}} \neq T_{\text{load}}$).
Fuel Control	No control over fuel or working fluid supply.	Directly adjusts fuel/steam flow via throttle valve.
Working Principle	Physical energy storage in kinetic form ($E = \frac{1}{2}I\omega^2$).	Force equilibrium (centrifugal force vs. gravity/spring).
Control Type	Open-loop passive inertia control.	Closed-loop active feedback control.
Mass & Inertia	Massive rim with high mass moment of inertia (I).	Lightweight mechanism with minimal inertia.
Duty Cycle	Operates continuously during every engine stroke.	Operates intermittently only when load changes.
Mounting	Direct rigid mounting on crankshaft.	Driven via gear train or belt transmission.

Lecture 40: 5.3 Comparison between Flywheel and Governor I

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit 5:** Flywheels and Governors
- **Lecture 40:** Comparative Analysis of Speed Control Devices
- **Department:** Mechanical Engineering Department

Lecture 40: 5.3 Comparison between Flywheel and Governor II

Executive Summary

In prime movers such as internal combustion engines and steam turbines, speed fluctuations occur due to two distinct causes: cyclic variation of output torque during a single cycle, and variation of external load over multiple cycles. Speed control requires two fundamentally different mechanisms: the **Flywheel** (a passive energy reservoir) and the **Governor** (an active regulating controller).

Functional Overview of a Flywheel I

Definition & Primary Purpose

A flywheel is a heavy rotating element attached to the crankshaft that acts as an inertial energy storage reservoir. It regulates cyclic speed variations within each operating cycle of an engine.

- **Energy Buffer:** Stores kinetic energy during periods of excess power generation ($T_{\text{engine}} > T_{\text{mean}}$) and releases it during periods of deficit ($T_{\text{engine}} < T_{\text{mean}}$).
- **Cyclic Smoothing:** Mitigates cyclic speed fluctuations resulting from the periodic variation of turning moment (torque) produced during working strokes.
- **Intra-cycle Regulation:** Maintains near-constant angular speed over a single revolution or cycle (e.g., suction, compression, expansion, exhaust strokes in IC engines).

Functional Overview of a Flywheel II

- **Load Limitation:** A flywheel **cannot** compensate for or control speed variations caused by changes in external engine load.

Functional Overview of a Governor I

Definition & Primary Purpose

A governor is a mechanical or electromechanical speed-regulating governor that controls mean engine speed over longer periods in response to variations in external load.

- **Fluid Quantity Regulation:** Directly adjusts the supply of working fluid (fuel-air mixture, steam, or gas) entering the engine via a throttle valve.
- **Mean Speed Control:** Maintains the mean speed of the engine within specified permissible limits under varying external load conditions.
- **Dynamic Feedback Control:** Operates dynamically by shifting its sleeve position whenever the external load changes and engine speed deviates from equilibrium.

Functional Overview of a Governor II

- **Cycle Limitation:** A governor has **no effect** on speed variations occurring within a single operating cycle of the engine.

Flywheel Mathematical Formulation I

Governing Equations

The dynamics of a flywheel depend on its mass moment of inertia (I) and the coefficient of fluctuation of speed (C_s).

- **Coefficient of Fluctuation of Speed (C_s):**

$$C_s = \frac{\omega_1 - \omega_2}{\omega} = \frac{2(\omega_1 - \omega_2)}{\omega_1 + \omega_2} \tag{1}$$

  where ω_1 and ω_2 are maximum and minimum angular speeds, and ω is mean angular speed.

- **Maximum Fluctuation of Energy (ΔE):**

$$\Delta E = E_{\text{max}} - E_{\text{min}} = I\omega^2 C_s = mk^2\omega^2 C_s \tag{2}$$

Flywheel Mathematical Formulation II

- **Rim Mass Calculation (Rim-type Flywheel):**

$$m = \frac{\Delta E}{R^2\omega^2 C_s} \tag{3}$$

  where R is mean rim radius and m is rim mass (neglecting web/arm inertia).

- **Design Insight:** Higher inertia (I) reduces C_s, yielding smoother machine output.

Governor Mathematical Formulation I

Governing Equations

Governor operation relies on centrifugal forces balancing deadweights or spring forces to maintain sleeve equilibrium.

- **Height of a Simple Watt Governor (h):**

$$h = \frac{g}{\omega^2} = \frac{895}{N^2} \text{ meters} \tag{4}$$

where N is engine speed in RPM (neglecting link mass and friction).

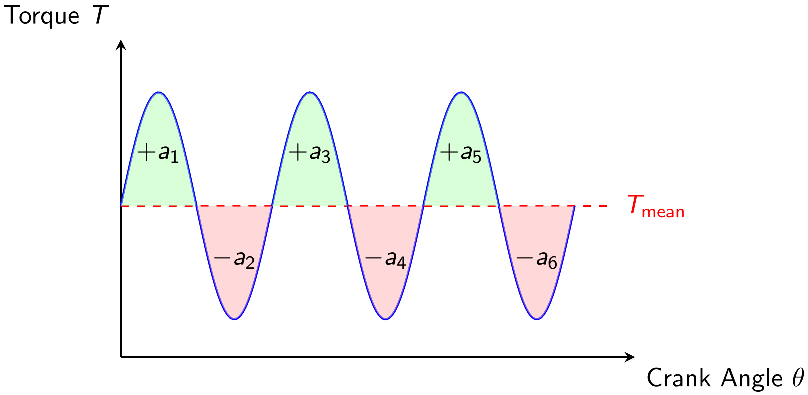
- **Sensitiveness of a Governor:**

$$\text{Sensitiveness} = \frac{N_1 - N_2}{N_{\text{mean}}} = \frac{2(N_1 - N_2)}{N_1 + N_2} \tag{5}$$

Governor Mathematical Formulation II

- **Hunting Phenomenon:** Occurs when a governor is excessively sensitive, causing continuous sleeve oscillation up and down without settling at equilibrium.
- **Isochronism:** A theoretical condition where equilibrium speed is constant for all radii of rotation ($N_1 = N_2$), giving infinite sensitiveness.

Flywheel Action: Turning Moment Diagram I

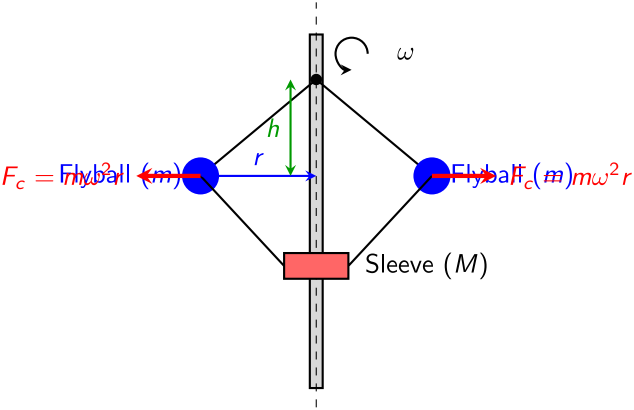


- **Green Shaded Areas ($+a_i$):** Engine torque exceeds mean resistance torque ($T > T_{\text{mean}}$); flywheel absorbs kinetic energy, increasing speed.

Flywheel Action: Turning Moment Diagram II

- **Red Shaded Areas ($-a_i$):** Engine torque drops below mean resistance torque ($T < T_{\text{mean}}$); flywheel releases kinetic energy, decreasing speed.
- **Extrema Points:** Engine speed reaches maximum at the end of positive energy loops and minimum at the end of negative energy loops.
- **Energy Fluctuation (ΔE):** Computed from maximum algebraic sum of loop areas, determining required flywheel mass.

Governor Action: Centrifugal Force Mechanism I



- **Speed Increase Step:** $\omega \uparrow \implies F_c = m\omega^2 r \uparrow \implies$ flyballs move outward ($r \uparrow$).

Governor Action: Centrifugal Force Mechanism II

- **Kinematic Response:** Outward expansion causes sleeve to lift upward, reducing height h.
- **Control Action:** Upward sleeve movement actuates throttle linkage, decreasing working fluid supply to lower engine speed back to ω_{mean}.
- **Speed Decrease Step:** $\omega \downarrow \implies F_c \downarrow \implies$ flyballs collapse inward, sleeve drops, opening throttle valve to restore speed.

Detailed Tabular Comparison: Flywheel vs. Governor I

Parameter	Flywheel	Governor
Primary Role	Controls cyclic speed variations within a single cycle	Regulates mean speed over long periods / multiple cycles
Cause of Fluctuation	Internal torque variations ($T(\theta)$) during working stroke	External load changes on the engine
Working Fluid Control	No influence on working fluid quantity	Directly controls working fluid via throttle valve
Operating Principle	Inertial energy storage ($E_k = \frac{1}{2}I\omega^2$)	Centrifugal / inertia force governor action
Action Type	Continuous, instantaneous, and passive	Intermittent, active, with finite response time
Key Metric	Coefficient of Fluctuation of Speed (C_s)	Sensitiveness, Stability, Isochronism
Typical Applications	IC Engines, Punching Presses, Shears	Turbines, IC Engines, Generators

Table: Comparative Matrix: Flywheel vs. Governor

Direct Comparison: Core Operational Differences I

Key Distinctions

Analyzing fundamental differences across five core operational dimensions.

1. **Nature of Regulation:** Flywheel operates intra-cycle (within a single cycle); Governor operates inter-cycle (across multiple revolutions under changing load).
2. **Energy Aspect:** Flywheel acts as a kinetic energy reservoir; Governor acts as an active fuel/energy supply controller.
3. **Influence on Working Fluid:** Flywheel has zero control over working fluid rate; Governor directly actuates throttle valve or fuel injection pump.
4. **Type of Fluctuation Targeted:** Flywheel addresses internal turning moment imbalance; Governor addresses external load-demand imbalance.

Direct Comparison: Core Operational Differences II

5. **Necessity by Machine Type:** Flywheel is compulsory in reciprocating IC engines and punch presses; Governor is compulsory for all regulated prime movers (turbines, generators).

Integration in Prime Movers and IC Engines I

Synergistic Co-operation

Both devices are essential in IC engines and operate simultaneously without interfering with each other's functions.

- **Complementary Operation:** The flywheel handles high-frequency cyclic torque spikes within a stroke, while the governor handles low-frequency load variations across cycles.
- **Passive vs. Active Control:** Flywheel action is passive and instantaneous (governed by Newton's laws: $T = I\alpha$), whereas governor action is active with finite measurement and actuation latency.

Integration in Prime Movers and IC Engines II

- **System Co-design:** Proper sizing requires selecting flywheel mass moment of inertia I to restrict C_s to acceptable limits (e.g., $C_s \leq 0.01$ for electric alternators), while tuning governor mass and spring stiffness to prevent hunting and maintain stability.

Summary of Comparative Metrics and Design Criteria I

Summary Metrics

- **Flywheel Performance Metric:** Coefficient of Fluctuation of Speed (C_s), Coefficient of Fluctuation of Energy (C_e).
- **Governor Performance Metrics:** Sensitiveness, Stability, Isochronism, Governor Effort, and Governor Power.

Core Rule of Thumb for GTU Examinations

- **Flywheel:** Smooths out the **turning moment curve** about the mean torque line.
- **Governor:** Shifts the **mean speed line** itself when load changes.

6.1 Balancing of Masses: Introduction & Fundamentals I

Course Context

Course: Theory of Machines and Mechanisms (DI03000141)
Unit 6: Balancing and Vibrations

- **High-Speed Machinery Need:** Rotating machine elements (crankshafts, turbine rotors, flywheels) inherently experience unbalance due to material non-homogeneity, machining tolerances, or asymmetric geometry.
- **Unbalanced Forces:** Accelerating masses generate dynamic inertia forces and moments that transmit periodic loads to supporting bearings.
- **Primary Objectives of Mass Balancing:**
  - Eliminate or minimize dynamic bearing reactions.
  - Prevent destructive structural vibrations and mechanical noise.

6.1 Balancing of Masses: Introduction & Fundamentals II

- Reduce fatigue failure risks and extend operational lifespan.
- Ensure smooth dynamic performance at elevated angular velocities.

The Mechanics of Rotating Unbalance I

Centrifugal Force Model

When a mass m rotates at a constant angular velocity ω at an eccentricity radius r from the axis of rotation, it experiences a radial centrifugal force:

$$F_c = m \cdot \omega^2 \cdot r$$

- **Directional Variation:** The direction of vector $\vec{F}_c$ continuously rotates with position angle $\theta = \omega t$, creating a harmonic dynamic load on the bearings.
- **Velocity Dependence:** The magnitude of F_c scales quadratically with speed (ω^2). Doubling rotor speed increases unbalanced forces by a factor of four.
- **Adverse Consequences of Unbalance:**

- Accelerated wear and premature failure of journal and rolling-element bearings.
- Transmitted foundation vibration endangering adjacent equipment.
- Shaft deflection, dynamic instability, and critical speed resonance excitation.

Static Balancing in a Single Plane I

Definition

A system of rotating masses is in **static balance** if the center of gravity (CG) of the entire system lies precisely on the geometric axis of rotation.

- **Equilibrium Condition:** The vector sum of all centrifugal forces acting in the plane must equal zero:

$$\sum \vec{F}_c = 0 \implies \omega^2 \sum m_i \vec{r}_i = 0 \implies \sum m_i \vec{r}_i = 0$$

- **Component Resolution:**

$$\sum m_i r_i \cos\theta_i = 0 \quad \text{and} \quad \sum m_i r_i \sin\theta_i = 0$$

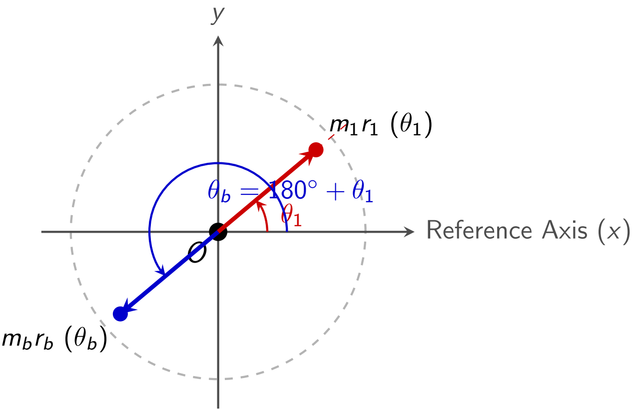
Static Balancing in a Single Plane II

- **Balancing Correction:** A single balancing mass m_b placed at radius r_b can counter an unbalance $m_1 r_1$ if:

$$m_b r_b = m_1 r_1 \quad \text{at angle} \quad \theta_b = \theta_1 + 180^\circ$$

- **Physical Test:** Under gravity alone, a statically balanced rotor placed on smooth horizontal knife-edge rails remains stationary at any angular orientation.

Vector Representation of Static Balancing I



- Out-of-balance vector $\vec{R_1} = m_1 r_1 \angle \theta_1$ generates radial force $m_1 r_1 \omega^2$.

Vector Representation of Static Balancing II

- Balancing mass vector $\vec{R}_b = m_b r_b \angle \theta_b$ acts diametrically opposite.
- Closed force polygon requires $\vec{R}_1 + \vec{R}_b = 0$.

Balancing of Several Masses in a Single Plane I

Analytical Solution Procedure

When multiple disturbing masses $m_1, m_2, \ldots, m_n$ rotate at radii $r_1, r_2, \ldots, r_n$ and angular positions $\theta_1, \theta_2, \ldots, \theta_n$:

1. Resolve centrifugal force products into orthogonal components:

$$H = \sum_{i=1}^{n} m_i r_i \cos\theta_i, \qquad V = \sum_{i=1}^{n} m_i r_i \sin\theta_i$$

2. Calculate the resultant unbalance force product R:

$$R = \sqrt{H^2 + V^2}$$

3. Determine the direction angle θ_R of the resultant force:

$$\theta_R = \tan^{-1}\left(\frac{V}{H}\right)$$

Balancing of Several Masses in a Single Plane II

4. Calculate balancing mass m_b at chosen radius r_b:

$$m_b r_b = R \implies m_b = \frac{R}{r_b}$$

5. Position balancing mass at angle: $\theta_b = \theta_R + 180^\circ$.

Dynamic Balancing (Multi-Plane Balancing) I

Why Static Balancing Is Insufficient

If unbalanced masses reside in different transverse planes along the shaft axis, static balance alone leaves unbalanced **turning couples**. These couples create dynamic bearing reactions during rotation.

- **Conditions for Complete Dynamic Equilibrium:**
  1. **Force Equilibrium (Static Balance):** Vector sum of all centrifugal forces must be zero:

  $$\sum \vec{F}_c = 0$$

  2. **Couple Equilibrium (Dynamic Balance):** Vector sum of moments of all centrifugal forces about any arbitrary reference plane must be zero:

  $$\sum \vec{M} = 0$$

Dynamic Balancing (Multi-Plane Balancing) II

- **Minimum Correction Planes:** Complete dynamic balancing of a rigid rotor requires at least **two distinct correction planes**.

Mathematical Formulation of Dynamic Balance I

Reference Plane Method

Select two reference planes, Plane A and Plane B, separated axially along the shaft.

- **Couple Equation about Plane A:** Taking moments of all centrifugal force products about Reference Plane A:

$$\sum_i (m_i r_i \cdot l_i) + (m_B r_B \cdot l_B) = 0$$

where l_i and l_B are signed axial distances from Plane A. Plane A's own moment arm is zero ($l_A = 0$), eliminating $m_A r_A$ from the couple equation.

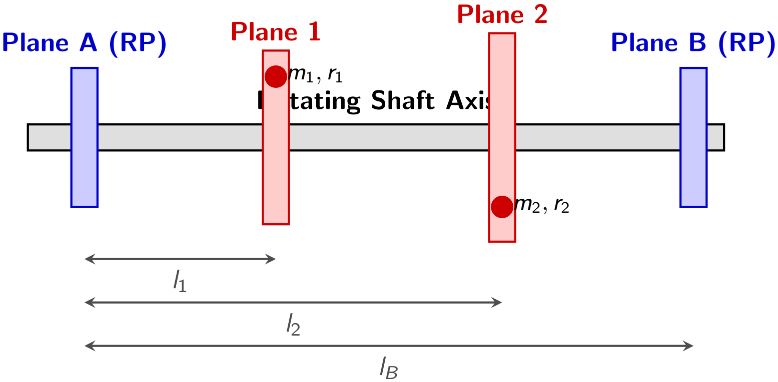
Mathematical Formulation of Dynamic Balance II

- **Force Equation for Net System:** Once $m_B r_B$ and its angle θ_B are calculated from the couple polygon:

$$m_A r_A + m_B r_B + \sum_i m_i r_i = 0$$

  Solving the force polygon yields the required balancing mass $m_A r_A$ and its angle θ_A.

Shaft and Planes Layout for Dynamic Balancing I



- Planes A and B serve as correction planes for placing counterweights.
- Axial distances to the right of RP A are positive ($+l$); distances to the left are negative ($-l$).

Tabular Method for Multi-Plane Balancing I

Systematic Analysis Table

Tabulating system values prevents sign errors and simplifies both analytical calculations and graphical vector construction.

Plane & **Mass** & **Radius** & **Force** $/\omega^2$ & **Distance** & **Couple** $/\omega^2$ & **Angle**
& (m) & (r) & $(m \cdot r)$ & (l) & $(m \cdot r \cdot l)$ & (θ)
Plane A (RP) & m_A & r_A & $m_A r_A$ & 0 & 0 & θ_A
Plane 1 & m_1 & r_1 & $m_1 r_1$ & $+l_1$ & $m_1 r_1 l_1$ & θ_1
Plane 2 & m_2 & r_2 & $m_2 r_2$ & $+l_2$ & $m_2 r_2 l_2$ & θ_2
Plane B (RP) & m_B & r_B & $m_B r_B$ & $+l_B$ & $m_B r_B l_B$ & θ_B

- **Step 1:** Construct Couple Polygon $(m \cdot r \cdot l)$ to find vector $m_B r_B l_B$ and angle θ_B.
- **Step 2:** Construct Force Polygon $(m \cdot r)$ to determine vector $m_A r_A$ and angle θ_A.

Industrial Applications and ISO 1940 Standards I

Industrial Applications

Dynamic balancing is essential across automotive, aerospace, and energy sectors: IC engine crankshafts, jet engine turbines, electric motor rotors, grinding wheels, and pump impellers.

- **Balancing Machines:**
  - **Hard-bearing machines:** Measure force reactions directly on rigid bearing supports.
  - **Soft-bearing machines:** Measure vibration displacement amplitudes on flexible supports.
- **ISO 1940 Balance Quality Grades (G-Grades):**
  - Represents maximum permissible residual specific unbalance $e_{\text{per}} \cdot \omega$ in mm/s.
  - **G40:** Automobile wheels, drive shafts.
  - **G6.3:** General machinery, pump impellers, electric armatures.
  - **G2.5:** Gas & steam turbines, high-speed machine tool drives.
  - **G1.0:** High-precision grinder spindles, tape recorder drives.

Summary and Key Governing Equations I

Summary of Governing Equations

- **Centrifugal Force Magnitude:**

$$F_c = m \cdot \omega^2 \cdot r$$

- **Static Equilibrium (Single Plane / Multi-Plane):**

$$\sum m_i r_i \cos\theta_i = 0 \quad \text{and} \quad \sum m_i r_i \sin\theta_i = 0$$

- **Dynamic Equilibrium (Multi-Plane):**

$$\sum \vec{F_c} = 0 \quad \text{(Net Force)} \qquad \text{and} \qquad \sum \vec{M} = 0 \quad \text{(Net Couple)}$$

- **Balancing Mass Vector Condition:**

$$\vec{R_b} = -\vec{R}_{\text{unbalance}}, \qquad \theta_b = \theta_R + 180^\circ$$

Summary and Key Governing Equations II

- **Key Takeaway:** Static balance guarantees force equilibrium; dynamic balance requires simultaneous force and couple equilibrium across at least two correction planes.

Lecture 42: Course Overview & Objectives I

- **Course**: Theory of Machines and Mechanisms (GTU Code: DI03000141)
- **Unit**: Balancing and Vibrations
- **Topic**: 6.1 Balancing of Masses
- **Instructor**: Mechanical Engineering Department

Lecture 42: Course Overview & Objectives II

Primary Learning Objectives

- Understand the origin, mechanics, and physical effects of unbalance in rotating machinery.
- Formulate analytical and graphical equilibrium conditions for static and dynamic balancing.
- Analyze single-plane and multi-plane rotating mass systems using force and couple vectors.
- Apply reference plane techniques to determine balancing mass magnitude, radius, and angular location.

Introduction to Balancing of Rotating Masses I

- **Concept of Unbalance**: When a mass m rotates in a circular path of radius r at angular velocity ω, it experiences centripetal acceleration, resulting in an outward dynamic reaction force on the shaft bearings called **centrifugal force**:

$$F_c = m \cdot \omega^2 \cdot r$$

- **Root Causes of Unbalance**:
  - Manufacturing tolerances, dimensional errors, and casting blowholes.
  - Non-uniform mass density and material inhomogeneities.
  - Asymmetrical rotor geometry, keyways, and eccentric shaft machining.
- **Adverse Mechanical Effects**:
  - Severe dynamic loads on supporting shaft bearings.
  - Excessive structural vibration and noise transmission.
  - Accelerated fatigue failure, reduced machine lifespan, and operational hazard.

Static Balancing vs. Dynamic Balancing I

Table: Comparison Between Static and Dynamic Balancing

Parameter & **Static Balancing** & **Dynamic Balancing**
Primary Condition & Center of gravity lies on & Axis of rotation coincides with & the rotation axis ($\sum \vec{F} = 0$) & principal axis of inertia ($\sum \vec{F} = 0, \sum \vec{M} = 0$)
Planes Required & Single balancing plane & At least two different balancing & is sufficient & transverse planes
Testing Mode & Static / stationary condition & Dynamic condition at actual & on frictionless rails & operational rotating speed
Couple Action & Ignores couples acting along & Completely eliminates net dynamic & the length of the shaft & couple (bending moment)

Static Balancing vs. Dynamic Balancing II

Key Relationship

Static balance is a *necessary but not sufficient* condition for dynamic balance. A dynamically balanced rotor is always statically balanced, but a statically balanced rotor may still be dynamically unbalanced.

Balancing of a Single Rotating Mass I

- **System Description**: Consider a single disturbing mass m_1 attached to a shaft at radius r_1, rotating at uniform angular speed ω.
- **Centrifugal Force Generated**:

$$F_{c1} = m_1 \cdot \omega^2 \cdot r_1$$

- **Balancing Strategy**: To eliminate bearing reaction forces, introduce a single balancing mass m_b at radius r_b in the same transverse plane, oriented directly opposite (180°) to m_1.
- **Equilibrium Condition**:

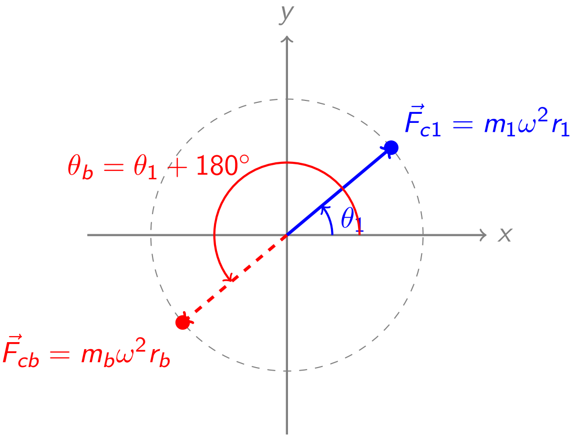
$$F_{cb} = F_{c1} \implies m_b \cdot \omega^2 \cdot r_b = m_1 \cdot \omega^2 \cdot r_1$$

$$m_b \cdot r_b = m_1 \cdot r_1 \implies m_b = \frac{m_1 \cdot r_1}{r_b}$$

- **Angular Position**:

$$\theta_b = \theta_1 + 180^\circ$$

Vector Representation of Single Mass Balancing I



- The disturbing centrifugal force vector $\vec{F}_{c1}$ acts outwards at angle θ_1.

Vector Representation of Single Mass Balancing II

- The counter-balancing vector $\vec{F}_{cb}$ acts in the exact opposite direction $(\theta_1 + 180^\circ)$.
- Resultant dynamic force on the shaft axis: $\vec{F}_{c1} + \vec{F}_{cb} = 0$.

Several Masses Rotating in a Single Plane: Analytical Method I

- **Problem Formulation**: Given n disturbing masses $m_1, m_2, \ldots, m_n$ rotating at radii $r_1, r_2, \ldots, r_n$ at angular positions $\theta_1, \theta_2, \ldots, \theta_n$ in a single plane.
- **Resolution into Rectangular Components**:

$$\sum(mr)_x\& = \sum_{i=1}^{n} m_i r_i \cos\theta_i = m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + \ldots$$

$$\sum(mr)_y\& = \sum_{i=1}^{n} m_i r_i \sin\theta_i = m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + \ldots$$

Several Masses Rotating in a Single Plane: Analytical Method II

- **Resultant Unbalanced Centrifugal Vector (R)**:

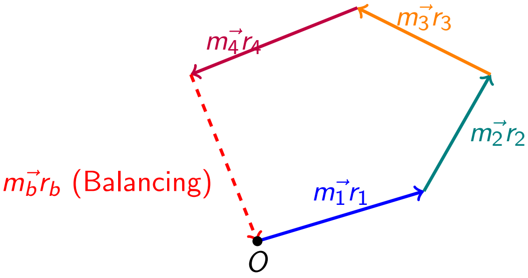
$$R = \sqrt{\left(\sum (mr)_x\right)^2 + \left(\sum (mr)_y\right)^2}$$

$$\theta_R = \tan^{-1}\left(\frac{\sum (mr)_y}{\sum (mr)_x}\right)$$

- **Balancing Mass Parameters**:

$$m_b \cdot r_b = R \quad \text{acting at angle} \quad \theta_b = \theta_R + 180^\circ$$

Force Polygon Method for Multiple Masses I



- **Graphical Procedure**:
  1. Select a suitable vector scale for mass-radius products (mr).
  2. Draw vector $\vec{m_1 r_1}$ parallel to θ_1, followed by $\vec{m_2 r_2}$ at θ_2 tip-to-tail, up to $\vec{m_n r_n}$.
  3. The closing vector from the terminus of the last vector back to origin O represents $\vec{m_b r_b}$.
  4. Measure the length to find magnitude $m_b r_b$ and angle for θ_b.

Masses Rotating in Different Planes: Forces and Couples I

- **Multi-Plane Shaft Kinematics**: When rotating masses lie in different transverse planes along the longitudinal shaft axis, centrifugal forces act along separate lines of action.
- **Induced Dynamics**:
  - Net centrifugal force vector causing radial shaking of bearings.
  - Net dynamic couple (moment) causing angular tipping/rocking of the shaft.
- **Conditions for Complete Dynamic Equilibrium**:
  1. **Primary Force Balance**: Vector sum of all centrifugal forces must be zero:

$$\sum \vec{F}_c = 0 \implies \sum m_i r_i \vec{e}_i = 0$$

Masses Rotating in Different Planes: Forces and Couples II

2. **Secondary Couple Balance**: Vector sum of all centrifugal couples about any reference plane must be zero:

$$\sum \vec{M}_c = 0 \implies \sum m_i r_i l_i \vec{e}_i = 0$$

- **Requirement**: Dynamic balance requires at least **two balancing planes**.

Tabular Method for Multi-Plane Balancing I

- **Reference Plane (RP) Selection**: Select one of the balancing planes as RP1 to eliminate its unknown distance ($l = 0$) from the couple equations.

Table: Standard Tabular Layout for Multi-Plane Balancing Analysis

Plane & **Mass** (m) & **Radius** (r) & **Force**/ω^2 & **Dist from RP1** & **Couple**/ω^2 & **Angle**
& (kg) & (m) & ($m \cdot r$) & (l, m) & ($m \cdot r \cdot l$) & (θ)
Plane 1 & m_1 & r_1 & $m_1 r_1$ & l_1 & $m_1 r_1 l_1$ & θ_1
RP 1 (B_1) & m_{b1} & r_{b1} & $m_{b1} r_{b1}$ & 0 & 0 & θ_{b1}
Plane 2 & m_2 & r_2 & $m_2 r_2$ & l_2 & $m_2 r_2 l_2$ & θ_2
RP 2 (B_2) & m_{b2} & r_{b2} & $m_{b2} r_{b2}$ & l_{b2} & $m_{b2} r_{b2} l_{b2}$ & θ_{b2}

- Distances l_i to the right of RP1 are positive ($+$); distances to the left are negative ($-$).

Couple and Force Polygons for Multi-Plane Systems

- **Step 1: Solve Couple Polygon First**:
  - Resolve couples about RP1:

$$\sum(mrl)_x = \sum m_i r_i l_i \cos\theta_i = 0, \quad \sum(mrl)_y = \sum m_i r_i l_i \sin\theta_i = 0$$

  - Draw the Couple Polygon using known vectors. The closing vector gives the couple required at RP2: $m_{b2} r_{b2} l_{b2}$.
  - Compute magnitude m_{b2} and angle θ_{b2}.
- **Step 2: Solve Force Polygon Second**:
  - Resolve forces including all known masses and the newly solved $m_{b2} r_{b2}$:

$$\sum(mr)_x = 0, \quad \sum(mr)_y = 0$$

  - The closing vector of the force polygon determines $m_{b1} r_{b1}$ and angle θ_{b1} for RP1.

Dynamic Balancing Machines and Industrial Applications I

- **Dynamic Balancing Equipment**:
  - **Hard-Bearing Machines**: Stiff supports; force sensors directly measure dynamic bearing reaction forces during rotation.
  - **Soft-Bearing Machines**: Flexible supports; measure displacement amplitudes of shaft vibration at resonance.
  - **Instrumentation**: Optical phase encoders and piezoelectric accelerometer sensors.
- **Key Engineering Applications**:
  - High-speed steam, gas, and jet engine turbine rotors.
  - Internal combustion engine crankshafts and flywheels.
  - Automobile wheel assemblies, driveshafts, and prop shafts.
  - Industrial centrifuges, blowers, and electric motor armatures.

Summary and Key Engineering Takeaways I

- **Fundamental Principles**:
  - Unbalanced centrifugal force: $F_c = m\omega^2 r$.
  - Static balance requires $\sum \vec{F} = 0$; dynamic balance requires $\sum \vec{F} = 0$ AND $\sum \vec{M} = 0$.
- **Design Guidelines for Engineers**:
  - Single plane rotating unbalance can be balanced by one mass in the same plane.
  - Multi-plane unbalance requires at least **two balancing planes**.
  - Always construct/solve the **Couple Polygon first** to isolate one unknown, followed by the **Force Polygon second**.
  - Proper balancing eliminates bearing fatigue, reduces vibration noise, and ensures operational reliability in turbomachinery.

Introduction to Balancing of Masses I

Course Information

Course: Theory of Machines and Mechanisms (DI03000141)
Topic: 6.1 Balancing of Masses (Static and Dynamic Balancing)

Motivation & Problem Statement

- High-speed machinery elements (crankshafts, turbine rotors, flywheels, pump impellers) often possess asymmetrical mass distributions due to manufacturing tolerances, material non-homogeneity, or operational design.
- When rotated, eccentric masses produce out-of-balance centrifugal forces.
- Unbalanced dynamic forces induce unwanted structural vibrations, excessive bearing wear, acoustic noise, dynamic stresses, and potential catastrophic dynamic failure.

Introduction to Balancing of Masses II

Core Objective of Balancing

The primary goal of balancing is to redistribute masses or attach corrective counter-masses such that net dynamic forces and net dynamic couples acting on the rotating system are reduced to acceptable engineering limits.

The Physics of Unbalance I

Centrifugal Force Formulation

Consider a mass m rotating at an angular velocity ω (rad/s) at a radial distance (eccentricity) r from the axis of rotation:

$$F_c = mr\omega^2$$

- **Direction:** Radially outward from the center of rotation along the line connecting the rotation axis to the mass center.
- **Dynamic Behavior:** As the shaft rotates, the direction of F_c rotates continuously at angular speed ω.
- **Speed Dependence:** $F_c \propto \omega^2$; doubling the rotational speed quadruples the unbalanced dynamic load acting on the bearings.

The Physics of Unbalance II

Consequences of Dynamic Unbalance

Transmitted cyclic loads cause fatigue failure in bearings, dynamic foundation vibration, shaft deflection, noise, and destabilizing shaft whirl phenomena.

Static vs. Dynamic Balancing

Fundamental Definitions

- **Static Balancing:** A system is statically balanced if the combined mass center (center of gravity) lies exactly on the axis of rotation. The net dynamic force is zero.
- **Dynamic Balancing:** A system is dynamically balanced if the principal axis of inertia coincides with the axis of rotation. The net dynamic force AND net dynamic couple are both zero.

Table: Comparison of Static and Dynamic Balancing

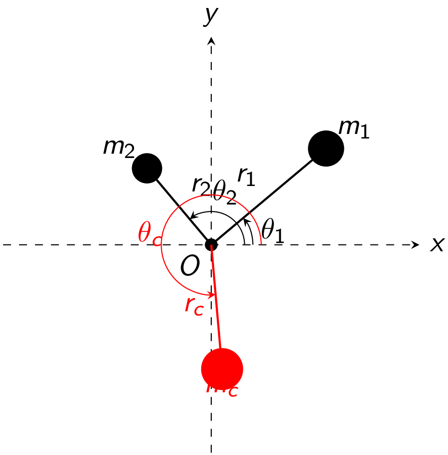
Parameter & Static Balancing & Dynamic Balancing
Mass Distribution & Masses rotate in a single transverse plane. & Masses rotate in multiple transverse planes along a shaft.
Equilibrium Condition & Force vector sum is zero: $\sum \vec{F}_c = 0 \implies \sum m_i \vec{r}_i = 0$. & Both force and couple vector sums are zero: $\sum \vec{F} = 0$ and $\sum \vec{C} = 0$.
Static State & System remains stationary in any angular position. & Requires rotation to detect unbalanced couple moments.
Balancing Planes Required & Single plane is sufficient. & Minimum of two distinct planes required.

Static Balancing in a Single Plane: Geometry I

Vector Diagram of Coplanar Masses

When several masses $m_1, m_2, \ldots, m_n$ rotate in a single plane at radii $r_1, r_2, \ldots, r_n$ and angular positions $\theta_1, \theta_2, \ldots, \theta_n$, they generate individual centrifugal forces acting outward.

Static Balancing in a Single Plane: Geometry II



Resolution of Centrifugal Forces

Since angular velocity ω is common to all rotating masses, the centrifugal force proportional terms $m_i r_i$ can be resolved along orthogonal x and y axes:

$$\sum H\& = \sum_{i=1}^{n} m_i r_i \cos\theta_i = m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + \ldots$$

$$\sum V\& = \sum_{i=1}^{n} m_i r_i \sin\theta_i = m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + \ldots$$

Analytical Method for Single Plane Balancing II

Resultant Force and Counter-Mass Determination

- **Resultant Unbalanced Mass-Radius Product (R):**

$$R = m_c r_c = \sqrt{\left(\sum H\right)^2 + \left(\sum V\right)^2}$$

- **Direction of Resultant Vector (θ_R):**

$$\tan \theta_R = \frac{\sum V}{\sum H}$$

- **Balancing Mass Position (θ_c):** The balancing mass m_c placed at radius r_c must produce a force equal and opposite to R:

$$\theta_c = \theta_R + 180^\circ$$

Graphical Method: Force Polygon I

Graphical Method: Force Polygon II

Constructing the Force Polygon

1. **Space Diagram:** Draw a space diagram showing the true angular positions $\theta_1, \theta_2, \ldots, \theta_n$ of all given masses relative to a reference horizontal line.
2. **Vector Scale:** Choose a suitable scale for mass-radius products $m_i r_i$ (e.g., $1\text{ cm} = 10\text{ kg}\cdot\text{mm}$).
3. **Sequential Vector Addition:**
   - Draw vector ab parallel to line of mass m_1 representing magnitude $m_1 r_1$.
   - From point b, draw vector bc parallel to mass m_2 representing magnitude $m_2 r_2$.
   - Repeat for all n masses until final vector point k is reached.
4. **Closing Vector:** If the polygon does not close, vector ka (from end point to start point) represents the balancing mass-radius product $m_c r_c$ in magnitude and direction.

Dynamic Unbalance and Couple Formation I

Why Multi-Plane Systems Require Dynamic Balancing

- When masses rotate in different planes along the length of a shaft, their centrifugal forces act in parallel offset planes.
- Even if the force polygon closes ($\sum \vec{F} = 0$, achieving static balance), the equal and opposite forces in separate planes create turning moments or **unbalanced couples**.
- These couples cause the shaft axis to rock relative to its bearing support lines, creating dynamic bearing loads.

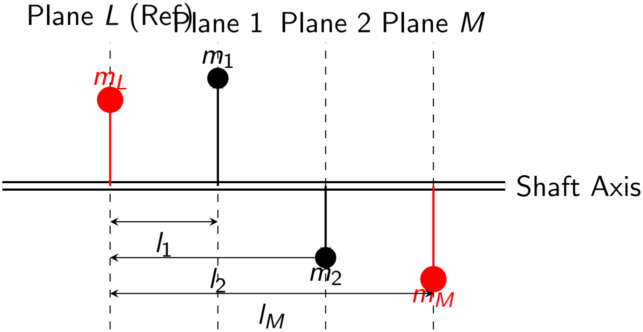
Two-Plane Balancing Rule

To completely balance any arbitrary distribution of rotating masses along a longitudinal axis, at least **two balancing masses** placed in **two distinct reference planes** are required.

Dynamic Balancing: Multi-Plane Configuration I

Schematic of Multi-Plane Rotating System

Consider masses m_1, m_2 in planes 1 and 2, balanced by counter-masses m_L, m_M in reference planes L and M.



The Tabular Method for Dynamic Balancing I

Tabular Layout

Set up a structured calculation table by choosing one balancing plane as the **Reference Plane (RP)**. Distances l measured to the right of RP are positive, and to the left are negative.

Table: Dynamic Balancing Calculation Table Format

Plane & **Mass** (m) & **Radius** (r) & **Force** (mr) & **Dist.** l & **Couple** (mrl) & **Angle** (θ)
L (RP) & m_L & r_L & $m_L r_L$ & 0 & 0 & θ_L
1 & m_1 & r_1 & $m_1 r_1$ & $+l_1$ & $m_1 r_1 l_1$ & θ_1
2 & m_2 & r_2 & $m_2 r_2$ & $+l_2$ & $m_2 r_2 l_2$ & θ_2
M & m_M & r_M & $m_M r_M$ & $+l_M$ & $m_M r_M l_M$ & θ_M

Procedure for Dynamic Balancing Analysis I

Step-by-Step Analytical/Graphical Solution

1. **Couple Polygon / Equations:** Compute components of couples relative to RP:

$$\sum C_x = \sum m_i r_i l_i \cos\theta_i, \quad \sum C_y = \sum m_i r_i l_i \sin\theta_i$$

Since $l_L = 0$, the couple due to m_L is zero. Solving the couple equilibrium equation determines $m_M r_M l_M$ and its direction θ_M.

2. **Force Polygon / Equations:** With m_M known, calculate force components:

$$\sum H = \sum m_i r_i \cos\theta_i, \quad \sum V = \sum m_i r_i \sin\theta_i$$

Solving the force equilibrium equation determines the remaining unknown balancing parameter $m_L r_L$ and angle θ_L.

Practical Engineering Applications & Consequences I

Industrial Applications

Dynamic balancing is mandatory across modern mechanical engineering sectors:

- **Automotive Engines:** Crankshafts, flywheels, camshafts, drive shafts, and wheel-tire assemblies.
- **Turbomachinery:** Steam turbine rotors, gas turbine compressor disks, turbochargers.
- **Industrial Machinery:** Centrifugal pump impellers, electric motor armatures, fan blowers, machine tool spindles.

Practical Engineering Applications & Consequences II

Modern Balancing Techniques

- **Hard-Bearing / Soft-Bearing Machines:** Measure dynamic bearing reactions using piezoelectric force transducers and optical position encoders.
- **In-Situ Field Balancing:** Uses portable vibration analyzer instrumentation to measure vibration amplitude and phase lag at operating speed.

Lecture Overview: Vibrations in Machines I

Course Information

- **Course:** Theory of Machines and Mechanisms (DI03000141)
- **Unit 6:** Balancing and Vibrations
- **Topic:** 6.2 Vibrations in Machines

Lecture Overview: Vibrations in Machines II

Learning Objectives

- Understand the fundamental physics, causes, and dynamic consequences of mechanical vibrations.
- Formulate governing equations of motion for Single Degree-of-Freedom (SDOF) mass-spring-damper systems.
- Analyze free vibration decay regimes and logarithmic decrement (δ).
- Evaluate force transmissibility (TR) and establish design parameters for vibration isolation.
- Investigate harmonic excitation from rotating unbalance, whirling of shafts, critical speeds, and seismic measurement instruments.

Fundamental Definition

Vibrations are periodic oscillatory motions of mechanical systems around their static equilibrium positions, governed by the continuous interchange of kinetic and potential energy under inertial, elastic, and damping forces.

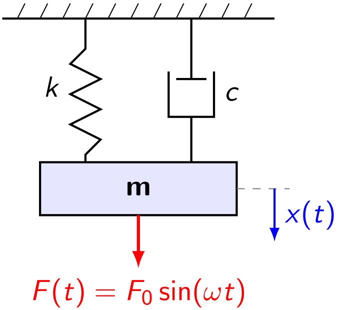
Primary Sources of Machine Vibrations

- **Inertial Unbalance:** Unbalanced mass distribution in high-speed rotating or reciprocating components.
- **Kinematic Misalignment:** Angular or parallel offset misalignment across coupled shafts.
- **Component Imperfections:** Pitch errors in gear meshes, bearing race defects, and belt drives.
- **Transient Impact Forces:** Reciprocating inertia forces, combustion pressure pulses, and cutting tool chatter.

Dynamic Consequences

Alternating cyclic stresses induce material fatigue failure, acoustic noise generation, precision loss, structural bolt loosening, and accelerated component wear.

Single Degree-of-Freedom (SDOF) Model I



Governing Equation of Motion

Applying D'Alembert's principle to the mass:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

where:

- m: System mass (kg)
- k: Equivalent spring stiffness (N/m)
- c: Viscous damping coefficient (N·s/m)

Single Degree-of-Freedom (SDOF) Model II

Fundamental System Parameters

- **Undamped Natural Frequency:** $\omega_n = \sqrt{\frac{k}{m}}$ (rad/s)
- **Critical Damping Coefficient:** $c_c = 2\sqrt{km} = 2m\omega_n$ (N·s/m)
- **Damping Ratio:** $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$

Free Vibration Analysis of SDOF Systems I

Homogeneous Differential Equation ($F(t) = 0$)

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

Classification of Damping Regimes

Regime	**Condition**	**Dynamic Behavior & Response Characteristics**
Underdamped	$\zeta < 1$	Oscillatory exponential decay at frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Characteristic of real mechanical structures.
Critically Damped	$\zeta = 1$	Non-oscillatory rapid decay back to equilibrium in minimal time. Essential for recoil & instrument design.
Overdamped	$\zeta > 1$	Slow exponential creep back to static position without overshoot. High energy dissipation rate.

Free Vibration Analysis of SDOF Systems II

Logarithmic Decrement (δ)

Quantifies damping from consecutive peak displacement amplitudes (x_1, x_2):

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta \quad (\text{for } \zeta \ll 1)$$

Force Transmissibility and Isolation I

Concept of Vibration Isolation

Vibration isolation aims to suppress force transmission between a vibrating machine base and its supporting foundation using resilient spring-damper mounts.

Transmissibility Ratio (TR)

The ratio of dynamic force transmitted to the foundation (F_t) to the excitation force amplitude (F_0):

$$TR = \frac{F_t}{F_0} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

where $r = \frac{\omega}{\omega_n}$ is the operating frequency ratio.

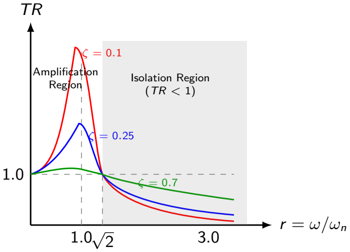
Force Transmissibility and Isolation II

Transmitted Force Vector Composition

The force transmitted to the foundation is the resultant vector sum of spring stiffness force and viscous damping force:

$$F_t = X\sqrt{k^2 + (c\omega)^2} = kX\sqrt{1 + (2\zeta r)^2}$$

Force Transmissibility Characteristics



Key Characteristics

- **Crossover Point:** At $r = \sqrt{2}$, $TR = 1$ regardless of damping ratio ζ.
- **Isolation Region ($r > \sqrt{2}$):** Effective isolation requires $r > \sqrt{2}$ to achieve $TR < 1$.
- **Damping Paradox:** For $r > \sqrt{2}$, higher damping ζ *increases* force transmission!

Design Compromise

Damping is required to limit resonance amplitude during startup and shutdown transitions through $r = 1$, while low stiffness ensures operating speed remains well within $r > \sqrt{2}$.

Harmonic Excitation: Rotating Unbalance I

Excitation Mechanism

Unbalance occurs when the mass center of a rotor of mass M is offset by eccentricity e, yielding an equivalent unbalanced mass m_0 rotating at speed ω:

$$F(t) = m_0 e \omega^2 \sin(\omega t)$$

Steady-State Amplitude Response

Solving the SDOF equation under harmonic excitation yields vibration amplitude X:

$$\frac{MX}{m_0 e} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Harmonic Excitation: Rotating Unbalance II

Speed-Dependent Operational Regimes

- **Low Speed ($r \ll 1$):** Centrifugal unbalance force is negligible; amplitude $X \to 0$.
- **Resonance ($r = 1$):** Peak vibration amplitude governed entirely by damping: $\frac{MX}{m_0 e} = \frac{1}{2\zeta}$.
- **High Speed ($r \gg 1$):** Amplitude asymptotes to constant limit: $X \to \frac{m_0 e}{M}$ (Rotor self-centers about mass axis).

Whirling of Shafts and Critical Speeds I

Definition of Shaft Whirling

Whirling is the rotational bending deflection of a shaft around its geometric bearing axis caused by centrifugal forces on eccentric rotor masses.

Analysis of Critical Speed

Consider a disc of mass m mounted on a elastic shaft of lateral stiffness k, with mass eccentricity e. At rotational speed ω:

$$m(y+e)\omega^2 = ky \implies y = \frac{e\omega^2}{\omega_n^2 - \omega^2} = \frac{er^2}{1-r^2}$$

where critical speed equals lateral natural frequency: $\omega_{cr} = \omega_n = \sqrt{\frac{k}{m}}$.

Whirling of Shafts and Critical Speeds II

Phase Relationship and Deflection Dynamics

- **Sub-critical ($r < 1$):** Deflection y is in-phase with e; heavy side lies on the outside of the shaft bend.
- **Super-critical ($r > 1$):** Deflection y is 180° out-of-phase with e; mass center shifts inward towards geometric center of rotation.

Vibration Measurement: Seismic Instruments I

Working Principle of Seismic Pickups

A seismic pickup consists of a spring-mass-damper system enclosed in a rigid frame attached to the vibrating surface $y(t) = Y \sin(\omega t)$. The relative displacement $z = x - y$ is converted into electrical signals.

Vibration Measurement: Seismic Instruments II

Instrument Classification Matrix

Design Feature	Vibrometer (Seismic)	Accelerometer
Measured Quantity	Base Displacement $y(t)$	Base Acceleration $\ddot{y}(t)$
Frequency Ratio	$\omega \gg \omega_n \quad (r \gg 1)$	$\omega \ll \omega_n \quad (r \ll 1)$
Spring Stiffness	Very soft spring (low ω_n)	Very stiff spring (high ω_n)
Seismic Mass	Heavy mass	Lightweight mass
Output Signal	$Z \approx Y$ (Seismic mass remains fixed)	$Z \approx \frac{\omega^2 Y}{\omega_n^2} \propto$ Acceleration
Transducer Type	LVDT, Optical proximity	Piezoelectric crystal (PZT)

Vibration Mitigation Strategies in Machines I

Engineering Strategies for Vibration Control

- **Source Reduction / Dynamic Balancing:** Field balancing per ISO 1940 standards; precision optical shaft alignment.
- **Stiffness & Mass Tuning:** Modifying structural stiffeners or mass distribution to shift natural frequency ω_n away from operating speeds ($\pm 20\%$ safety margin).
- **Dynamic Vibration Absorbers (Tuned Mass Dampers):** Attaching an auxiliary secondary mass-spring system (m_2, k_2) tuned such that $\omega_2 = \sqrt{k_2/m_2} = \omega$, creating anti-resonance to nullify primary system motion.
- **Damping Treatments:** Applying viscoelastic constrained-layer damping, squeeze-film dampers, or magnetorheological fluid mounts to absorb peak transient energy.

Summary of Key Formulas I

Essential Engineering Equations

Parameter / Quantity	**Mathematical Formula**
Undamped Natural Frequency	$\omega_n = \sqrt{\frac{k}{m}}$
Damping Ratio	$\zeta = \frac{c}{2\sqrt{km}}$
Damped Natural Frequency	$\omega_d = \omega_n\sqrt{1-\zeta^2}$
Logarithmic Decrement	$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\frac{x_1}{x_2}\right)$
Transmissibility Ratio	$TR = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$
Whirling Deflection Amplitude	$y = \frac{er^2}{1-r^2} \quad \left(r = \frac{\omega}{\omega_{cr}}\right)$

Course: DI03000141 – Theory of Machines and Mechanisms

Instructor: Professor of Mechanical Engineering

Lecture 45: Vibrations in Machines

Lecture Overview

- Introduction to mechanical vibrations in machine components.
- Understanding sources of machine excitation and dynamic responses.
- Fundamentals of modeling, analysis, and vibration mitigation.

Sources of Vibration in Machinery I

Machine vibrations are primarily caused by oscillating forces or displacement excitations resulting from design imperfections, wear, or operating conditions.

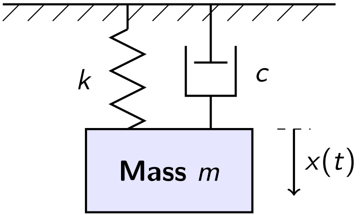
- **Unbalance in Rotating Elements:** Eccentricity of mass center relative to rotation axis creates a centrifugal force that varies as $F = m \cdot e \cdot \omega^2$.
- **Reciprocating Forces:** Acceleration and deceleration of pistons and slider-crank mechanisms generate primary and secondary inertial forces.
- **Misalignment of Couplings and Shafts:** Creates angular or offset deviations, inducing axial and radial forces at 1X and 2X rotational frequencies.

Sources of Vibration in Machinery II

- **Gear Mesh Excitation:** Transmission errors, tooth deflection under load, and wear cause periodic torque and force variations.
- **Aerodynamic and Hydrodynamic Forces:** Flow turbulence, vortex shedding, and cavitation in pumps, turbines, and fans.

Single-Degree-of-Freedom (SDOF) Model I

Mathematical representation of a machine mount showing basic elements: inertia (m), stiffness (k), and dissipation (c).



- Assumes rigid body mass moving in a single direction (vertical translation $x(t)$).
- Stiffness k represents structural elasticity or elastomeric mounts.
- Damping c represents viscous dissipation (oil film, internal material friction).

Governing Equation of Motion I

The dynamic behavior of the SDOF system is derived using Newton's Second Law or D'Alembert's Principle, leading to a second-order linear differential equation.

Equation of Motion

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

where $F(t)$ is the external exciting force.

- **Undamped Natural Frequency:**

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{(rad/s)}$$

representing the frequency of free vibration without damping.

- **Critical Damping Coefficient:**

$$c_c = 2\sqrt{km} = 2m\omega_n$$

  the minimum damping required for a non-oscillatory return to equilibrium.

- **Damping Ratio:**

$$\zeta = \frac{c}{c_c}$$

  classifies systems as underdamped ($\zeta < 1$), critically damped ($\zeta = 1$), or overdamped ($\zeta > 1$).

Free Vibration Response I

When a machine is disturbed from its equilibrium position and released, its transient response is determined by the damping ratio ζ.

- **Underdamped Response ($\zeta < 1$):** Decay is characterized by an exponential envelope $e^{-\zeta\omega_n t}$ multiplying a harmonic term $\cos(\omega_d t - \phi)$:

$$x(t) = Xe^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

- **Damped Natural Frequency:**

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

which is always lower than the undamped natural frequency.

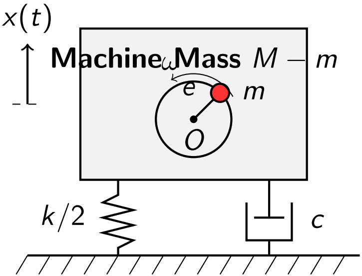
Free Vibration Response II

- **Logarithmic Decrement:**

$$\delta = \ln\left(\frac{x_i}{x_{i+1}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

  used to experimentally determine damping in machinery.
- **Overdamped and Critically Damped Systems:** Do not oscillate; critically damped systems return to equilibrium in the shortest time without overshoot.

A common vibration excitation in rotating machines where an eccentric mass m at distance e rotates at frequency ω.



- Vertical component of the centrifugal force is $F(t) = m \cdot e \cdot \omega^2 \sin(\omega t)$.

Rotating Unbalance Excitation II

- Steady-state amplitude X is given by:

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

where M is total mass and $r = \frac{\omega}{\omega_n}$ is the frequency ratio.

- At high speeds ($r \gg 1$), vibration amplitude asymptotically approaches $\frac{me}{M}$.

Forced Vibration & Resonance I

When subjected to a harmonic force $F_0 \sin(\omega t)$, the SDOF system exhibits a steady-state response at the excitation frequency.

- **Magnification Factor (MF):** Ratio of dynamic displacement amplitude to static displacement (F_0/k).

$$\mathrm{MF} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{where } r = \frac{\omega}{\omega_n}$$

- **Resonance:** Occurs when $r = 1$ (excitation frequency equals natural frequency), causing large amplitude oscillations limited only by damping.
- **Phase Angle (ϕ):** Phase lag between force and response. At resonance ($r = 1$), $\phi = 90^\circ$ regardless of damping.
- **Operating Regions:**
  - Sub-critical ($r < 1$, stiffness-dominated)
  - Resonant ($r \approx 1$, damping-dominated)
  - Super-critical ($r > 1$, inertia-dominated)

Vibration Isolation & Transmissibility I

Vibration isolation reduces the force transmitted from a vibrating machine to its foundation, or protects sensitive equipment from ground motion.

- **Transmissibility Ratio (TR):** Ratio of the force transmitted to foundation (F_t) to exciting force (F_0).

$$\mathrm{TR} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

- **Isolation Threshold:** Isolation is achieved only when $\mathrm{TR} < 1$, requiring frequency ratio $r > \sqrt{2}$.
- **Damping Trade-off:** Damping decreases peak transmissibility near resonance ($r < \sqrt{2}$), but increases transmissibility in the isolation region ($r > \sqrt{2}$).

- **Design Objective:** Select isolator stiffness k such that machine operating speed ω is at least 3 to 4 times natural frequency ω_n.

Whirling of Shafts & Critical Speeds I

Whirling is the rotation of the bent shaft about its longitudinal axis, occurring at specific speeds known as critical speeds.

- **Cause of Whirling:** Centrifugal force due to gravity deflection, manufacturing eccentricity, or rotor unbalance pulls the shaft outward.
- **Dynamic Deflection:**

$$y = e\frac{\omega^2}{\omega_n^2 - \omega^2} = e\frac{r^2}{1 - r^2}$$

  where e is initial center of mass eccentricity.
- **Critical Speed:** Rotational speed ω equal to lateral natural frequency of the shaft system. At this speed, deflection $y \to \infty$ (resonance).

Whirling of Shafts & Critical Speeds II

- **Self-Centering:** At speeds much higher than critical speed ($r \gg 1$), deflection $y \to -e$, meaning the shaft rotates about its center of mass.
- **Safe Operation:** Shafts must either run sub-critically ($r < 1$) or quickly pass through the critical speed to run super-critically ($r > 1$).

Engineering methods to manage and reduce machine vibrations fall into three categories: source reduction, path modification, and receiver isolation.

Table: Summary of Machine Vibration Control Techniques

Method & **Operating Principle** & **Typical Applications**
Rotor Balancing & Reduces mass eccentricity e (static/dynamic) & Turbines, motors, centrifugal pumps
Vibration Isolation & Decouples system using springs/elastomers ($r > \sqrt{2}$) & Machine foundations, HVAC units
Tuned Mass Damper & Auxiliary spring-mass tuned to cancel main vibration & Bridges, tall structures, engines
Active Control & Sensors and actuators generate counter-phase force & Precision optics, aerospace systems
Damping Treatment & Viscoelastic material absorbs energy & Engine covers, structural panels