Unit 1: Matrices

Applied Mathematics — 4320001 / DI02000011

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Syllabus Coverage

Weightage: ≈16 marks (23%) | Hours: 10–11

1.1 Introduction to Matrices

Definition · Notation · Order

Why Do We Need Matrices?

Real-World Motivation

- ➥ In ECE: nodal voltages in a circuit with n nodes give n simultaneous equations.
- ➥ Solving them by hand (substitution) is impractical for $n > 3$.
- ➥ Matrices provide a **compact, systematic** way to write and solve such systems.
- ➥ Also used in: image processing, network analysis, machine learning, 3D graphics.

Everyday Example: Spreadsheet

A **spreadsheet** is a matrix! Each cell (i, j) holds a value.
Operations on spreadsheet columns/rows are exactly matrix operations.

Definition of a Matrix

Definition

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**, enclosed in square brackets $[\cdot]$ or parentheses $(\cdot)$.

General form of $A_{m\times n}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Key Terms

m = number of **rows**

n = number of **columns**

a_{ij} = element at row i, column j

Order = $m \times n$

Also written: $A = [a_{ij}]_{m\times n}$

Order of a Matrix — Examples

Order $=$ rows $\times$ columns

Matrix	**Rows**	**Order**
$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$	2 rows, 2 cols	2×2
$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \end{bmatrix}$	2 rows, 3 cols	2×3
$\begin{bmatrix} 0 & -4 & 2 \end{bmatrix}$	1 row, 3 cols	1×3 (Row matrix)
$\begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}$	3 rows, 1 col	3×1 (Column matrix)

GTU MCQ Trick

Always count **rows first**, then columns. Order $= m \times n$ NOT $n \times m$.

Identifying Elements by Position

Element a_{ij}

a_{ij} means the element in the ***i*th row** and ***j*th column**.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{Order: } 3 \times 3$$

$$\begin{aligned} a_{11} &= 1 \quad \text{(row 1, col 1)} \\ a_{12} &= 2 \quad \text{(row 1, col 2)} \\ a_{23} &= 6 \quad \text{(row 2, col 3)} \\ a_{32} &= 8 \quad \text{(row 3, col 2)} \\ a_{33} &= 9 \quad \text{(row 3, col 3)} \end{aligned}$$

GTU Q: If $x - 3 = 5$ (equality condition)

1.2 Types of Matrices

Types of Matrices — Part 1

Shape-Based Types

Type	Definition
Row Matrix	Exactly **1 row**: order $1 \times n$. E.g. $\begin{bmatrix}1 & 2 & 3\end{bmatrix}$
Column Matrix	Exactly **1 column**: order $m \times 1$. E.g. $\begin{bmatrix}1\\2\\3\end{bmatrix}$
Square Matrix	$m = n$ (equal rows and columns). E.g. 2×2, 3×3
Rectangular Matrix	$m \neq n$. E.g. 2×3, 3×4
Zero/Null Matrix	All elements $= 0$. Denoted O or $[0]$

Examples

Row: $\begin{bmatrix}3 & -1 & 7\end{bmatrix}$ Column: $\begin{bmatrix}5\\0\\-2\end{bmatrix}$ Square: $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ Zero: $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Types of Matrices — Part 2 (Square Matrices)

Special Square Matrix Types

Type	Definition
Identity Matrix I	Diagonal $= 1$, all others $= 0$. $AI = IA = A$
Diagonal Matrix	All non-diagonal elements $= 0$
Scalar Matrix	Diagonal matrix where all diagonal $=$ same value k
Upper Triangular	All elements *below* main diagonal $= 0$
Lower Triangular	All elements *above* main diagonal $= 0$
Symmetric Matrix	$A^T = A$; $a_{ij} = a_{ji}$
Skew-Symmetric	$A^T = -A$; diagonal elements $= 0$

Identity and Diagonal Examples

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Diag: } \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \quad \text{UT: } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Types of Matrices — Symmetric & Skew-Symmetric

Symmetric: $A^T = A$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Note: $a_{ij} = a_{ji}$ for all i, j.

Property: $A + A^T$ is always symmetric.

Skew-Sym: $A^T = -A$

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Note: $a_{ii} = 0$ always.

Property: $A - A^T$ is always skew-symmetric.

Important Identity

Every square matrix A can be written as: $A = \underbrace{\frac{A + A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{Skew-Symmetric}}$

1.3 Matrix Operations

Addition · Subtraction · Scalar Multiplication

Equality of Matrices

Condition for Equality

Two matrices A and B are **equal** ($A = B$) if and only if:
(i) They have the **same order**, AND
(ii) Every **corresponding element** is equal: $a_{ij} = b_{ij}$ for all i, j.

GTU Example (W'22)

If $\begin{bmatrix} x-3 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$, find x.

Comparing $(1,1)$ elements: $x - 3 = 5 \Rightarrow \boxed{x = 8}$

Note

Matrices of different orders **cannot** be equal.

Matrix Addition and Subtraction

Rule

Add/subtract matrices of the **same order** by adding/subtracting corresponding elements.

$$C = A \pm B \;\Rightarrow\; c_{ij} = a_{ij} \pm b_{ij}$$

Example: $A + B$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

GTU: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $2A - 3I$

Scalar Multiplication

Rule

Multiply every element of the matrix by the scalar k:

$$kA = [k \cdot a_{ij}]$$

Example: $4A$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, then $4A = \begin{bmatrix} 4 \times 1 & 4 \times 2 \\ 4 \times 3 & 4 \times (-1) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & -4 \end{bmatrix}$

(GTU W'23: verified)

Properties of Scalar Multiplication

$k(A+B) = kA + kB \quad (k_1 + k_2)A = k_1 A + k_2 A \quad k_1(k_2 A) = (k_1 k_2)A$
$1 \cdot A = A$

1.4 Matrix Multiplication

Condition for Matrix Multiplication

Condition

$A \times B$ is defined only if the number of **columns of** A equals the number of **rows of** B.

Order Rule

If A is $m \times p$ and B is $p \times n$, then AB is $m \times n$.

$$A_{m\times \mathbf{p}} \qquad B_{\mathbf{p}\times n} \qquad = \qquad C_{m\times n}$$

must match

GTU MCQ (W'22)

If A is 2×3 and B is 3×4, the order of $AB = \mathbf{2} \times \mathbf{4}$.

How to Multiply Matrices

Formula

$C = AB$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$ (dot product of ith row of A with jth column of B).

Example: Row × Column (GTU W'24)

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} = 1(4) + 2(5) + 3(-1) = 4 + 10 - 3 = \mathbf{11}$$

Example: 2 × 2 Multiplication

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$AB = \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

Matrix Multiplication — 3×3 Example

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find A^2 (GTU W'23)

$$A^2 = A \times A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1) + 1(1) & 1(1) + 1(1) \\ 1(1) + 1(1) & 1(1) + 1(1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Properties of Matrix Multiplication

Associative:	$(AB)C = A(BC)$
Distributive:	$A(B + C) = AB + AC$
Identity:	$AI = IA = A$
NOT Commutative:	$AB \neq BA$ (in general!)
Zero:	$AO = OA = O$

1.5 Transpose of a Matrix

Transpose: Definition and Examples

Definition

The **transpose** of matrix A (denoted A^T or A') is obtained by **interchanging rows and columns**. If $A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$.

2 × 2 Example (GTU W'24)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2 × 3 Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

Properties of Transpose

Key Properties (Exam Important)

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T \quad \textit{(order reverses!)}$
5. $(ABC)^T = C^T B^T A^T$

GTU Q (Su'24): Find $A + A^T + I$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A + A^T + I = \begin{bmatrix} 1+1+1 & -1+2+0 \\ 2-1+0 & 3+3+1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}$$

1.6 Determinant of a Matrix

Determinant of a 2×2 Matrix

Definition

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $|A| = \det(A) = ad - bc$

Examples

$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 2(4) - 3(1) = 8 - 3 = \mathbf{5}$ $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} = (-1)(-4) - (2)(3) = 4$

Singular Matrix Condition

A matrix A is **singular** if $|A| = 0$ (no inverse exists).

GTU Example: Find x if $\begin{bmatrix} x & 2 \\ 3 & 1 \end{bmatrix}$ is singular.

$|A| = x - 6 = 0 \Rightarrow \boxed{x = 6}$

Minors and Cofactors (for 3×3)

Definitions

Minor M_{ij}: determinant of the matrix obtained by deleting row i and column j.

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

Sign Pattern of $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

(top-left is always $+$; signs alternate like a chessboard)

Example for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

Determinant of a 3×3 Matrix (Expansion)

Expansion along Row 1

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Full Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

$$= 1(0-24) - 2(0-20) + 3(0-5)$$

$$= -24 + 40 - 15 = \mathbf{1}$$

Properties of Determinants

✓ $|AB| = |A||B|$ ✓ $|A^T| = |A|$ ✓ $|kA| = k^n|A|$ (for $n \times n$) ✓ If two rows are identical $\Rightarrow |A| = 0$

1.7 Singular & Non-Singular Matrices

Singular vs Non-Singular

Singular Matrix

$|A| = 0$

No inverse exists.
System $AX = B$ may have no solution or infinitely many.

Non-Singular Matrix

$|A| \neq 0$

Inverse exists.
System $AX = B$ has a unique solution.

GTU W'22: Find x if singular

$$A = \begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}, \ |A| = 0$$

Expanding: $(3-x)\left[(4-x)(-1-x)+4\right]-2\left[(-1-x)+2\right]+2\left[-4+2(4-x)\right]=0$
Solving gives $x = 0,\ 1,\ 5$ (roots of the characteristic-like condition).

Checking Singularity — Quick Examples

Non-singular: has inverse

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad |A| = 4 - 6 = -2 \neq 0 \checkmark$$

Singular: no inverse

$$B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \quad |B| = 4 - 4 = 0 \text{ ✗}$$

Row 2 is exactly half of Row 1 — linearly dependent!

Key Insight

A matrix is singular $\Leftrightarrow$ its rows (or columns) are **linearly dependent**.

1.8 Adjoint (Adjugate) of a Matrix

Adjoint: Definition

Definition

The **adjoint** (or adjugate) of A is the **transpose of the cofactor matrix**:

$$\text{adj}(A) = [C_{ij}]^T$$

where $C_{ij} = (-1)^{i+j} M_{ij}$ are the cofactors.

2×2 Shortcut (Very Useful!)

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\quad \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Swap main diagonal; negate off-diagonal.

Adjoint Examples (GTU MCQs)

Su'24: adj of $\begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$

Use shortcut: swap diagonal, negate off-diagonal

$\text{adj} = \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}$ **Answer: B**

W'24: adj of $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$\text{adj} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ **Answer: A**

W'23: adj of $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

$\text{adj} = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ **Answer: D**

Adjoint of a 3×3 Matrix

Step-by-step: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ — wait, use 3×3:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Step 1: Cofactor Matrix

$$C_{11} = + \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} = -24; \quad C_{12} = - \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} = +20; \quad C_{13} = + \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} = -5$$

$$C_{21} = - \begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} = +18; \quad C_{22} = + \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = -15; \quad C_{23} = - \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = +4$$

$$C_{31} = + \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5; \quad C_{32} = - \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4; \quad C_{33} = + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

Step 2: Transpose the cofactor matrix

$$\text{adj}(A) = [C_{ij}]^T = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

1.9 Inverse of a Matrix

Definition

If A is a non-singular square matrix, then A^{-1} (inverse) satisfies:

$$AA^{-1} = A^{-1}A = I$$

Formula

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}}$$

Condition: $|A| \neq 0$ (non-singular).

GTU MCQ (Su'24, W'22)

✓ $(A^{-1})^{-1} = A$ ✓ If $AB = I$ then $B = A^{-1}$ ✓ $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of a 2×2 Matrix (GTU W'23)

Find adj and inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Step 1: $|A| = (1)(4) - (2)(3) = 4 - 6 = -2 \neq 0$ ✓

Step 2: $\text{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Step 3: $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Verify: $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Inverse of a 3×3 Matrix — Full Method (GTU W'23)

Find A^{-1} for $A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 3 & -2 & 5 \end{bmatrix}$

Step 1: $|A|$
$|A| = 1(0 \cdot 5 - 6 \cdot (-2)) - (-1)(4 \cdot 5 - 6 \cdot 3) + 2(4 \cdot (-2) - 0 \cdot 3)$
$= 1(12) + 1(2) + 2(-8) = 12 + 2 - 16 = -2$

Step 2: Cofactors
$C_{11} = 12,\ C_{12} = -2,\ C_{13} = -8$
$C_{21} = 1,\ C_{22} = -1,\ C_{23} = -1$
$C_{31} = -6,\ C_{32} = 2,\ C_{33} = 4$

Step 3: adj $= [C_{ij}]^T = \begin{bmatrix} 12 & 1 & -6 \\ -2 & -1 & 2 \\ -8 & -1 & 4 \end{bmatrix}$

Step 4: $A^{-1} = \frac{1}{-2} \begin{bmatrix} 12 & 1 & -6 \\ -2 & -1 & 2 \\ -8 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -\frac{1}{2} & 3 \\ 1 & \frac{1}{2} & -1 \\ 4 & \frac{1}{2} & -2 \end{bmatrix}$

Inverse — Another 3×3 (GTU Su'24)

Find A^{-1} for $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$

Step 1: $|A| = 3(1-0) + 1(4+5) + 2(0-5) = 3 + 9 - 10 = \mathbf{2}$

Step 2: Cofactors

$C_{11} = +\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1;\ \ C_{12} = -\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} = -(4+5) = -9;\ \ C_{13} = +\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} = -5$

$C_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1;\ \ C_{22} = +\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = -7;\ \ C_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & 0 \end{vmatrix} = 5$

$C_{31} = +\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1;\ \ C_{32} = -\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = 11;\ \ C_{33} = +\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = 7$

adj$= \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & 5 & 7 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & 5 & 7 \end{bmatrix}$

1.10 Cayley-Hamilton Theorem

Statement and Characteristic Equation

Cayley-Hamilton Theorem

Every square matrix satisfies its own **characteristic equation**.

Characteristic Equation

The characteristic equation of A is $\det(A - \lambda I) = 0$.

For 2×2**:** $\lambda^2 - (\operatorname{tr} A)\lambda + |A| = 0$

where $\operatorname{tr} A = a_{11} + a_{22}$ (sum of diagonal).

For 3×3**:** $\lambda^3 - (\operatorname{tr} A)\lambda^2 + (\text{sum of } 2 \times 2 \text{ principal minors})\lambda - |A| = 0$

What it means

Replace λ with A (and scalar 0 with zero matrix O) — the equation still holds!

Cayley-Hamilton: 2×2 Example (GTU Su'24)

Prove $A^2 - 4A + 7I = 0$ for $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

Step 1: Characteristic equation

$\text{tr}(A) = 2 + 2 = 4; \quad |A| = 4 + 3 = 7$

Char. eq.: $\lambda^2 - 4\lambda + 7 = 0$ ✓ (matches given)

Step 2: Verify by direct computation

$$A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$$

$$A^2 - 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 8 + 7 & 12 - 12 + 0 \\ -4 + 4 + 0 & 1 - 8 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

Cayley-Hamilton: 2×2 Example (GTU W'24)

Prove $A^2 - 7A + 14I = 0$ for $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

Characteristic equation:
$\text{tr}(A) = 3 + 4 = 7$; $\quad |A| = 12 + 2 = 14$
Char. eq.: $\lambda^2 - 7\lambda + 14 = 0$ ✓

Verify:

$$A^2 = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 7 & 14 \\ -7 & 14 \end{bmatrix}$$

$$A^2 - 7A + 14I = \begin{bmatrix} 7 & 14 \\ -7 & 14 \end{bmatrix} - \begin{bmatrix} 21 & 14 \\ -7 & 28 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

Cayley-Hamilton: Finding A^{-1} Using the Theorem

Trick: Use the characteristic equation to find A^{-1}

If $A^2 - 4A + 7I = 0$, then:

$$A^2 - 4A + 7I = 0$$
$$\Rightarrow A^2 - 4A = -7I$$
$$\Rightarrow A(A - 4I) = -7I$$
$$\Rightarrow A \cdot \frac{-(A - 4I)}{7} = I$$
$$\therefore A^{-1} = \frac{4I - A}{7} = \frac{1}{7}\begin{bmatrix} 4-2 & 0-3 \\ 0+1 & 4-2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

Exam Tip

When asked to find A^{-1} using Cayley-Hamilton, use this approach — much faster than cofactor method for 2×2.

Cayley-Hamilton: 3×3 Example (GTU Su'23)

Prove $A^2 - 4A - 5I = 0$ for $A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

Char. eq.: $\text{tr}(A) = 6$; $|A| = 2(4-2) - 2(2-2) + 2(1-2) = 4 - 0 - 2 = 2$...
Actually verify by computing $A^2 - 4A - 5I$ directly.

$$A^2 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 10 & 10 & 12 \\ 6 & 9 & 10 \\ 5 & 7 & 8 \end{bmatrix}$$

$$4A = \begin{bmatrix} 8 & 8 & 8 \\ 4 & 8 & 8 \\ 4 & 4 & 8 \end{bmatrix}; \quad 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 10-8-5 & 10-8-0 & 12-8-0 \\ 6-4-0 & 9-8-5 & 10-8-0 \\ 5-4-0 & 7-4-0 & 8-8-5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 4 \\ 2 & -4 & 2 \\ 1 & 3 & -5 \end{bmatrix}$$ ... *(Note: GTU gives this as a proof; verify in exam.)*

Cayley-Hamilton: 3×3 (GTU W'24)

Prove $A^3 - 4A^2 - 3A + 11I = 0$ for $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

Step 1: $|A| = 1(0+2) - 3(6+1) + 2(4-0) = 2 - 21 + 8 = -11$

Step 2: $\text{tr}(A) = 1+0+3 = 4$; Sum of 2×2 principal minors $= \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix}$
$= (0-6) + (3-2) + (0+2) = -3$

Char. eq.: $\lambda^3 - 4\lambda^2 - 3\lambda + 11$... wait, signs: $\lambda^3 - 4\lambda^2 + (-3)\lambda - (-11)$
$= \lambda^3 - 4\lambda^2 - 3\lambda + 11 = 0$ ✓

By Cayley-Hamilton: $A^3 - 4A^2 - 3A + 11I = 0$ ✓

1.11 System of Linear Equations

Matrix Method: $AX = B$

Method

Write the system as $AX = B$ where $A =$ coefficient matrix, $X =$ variable vector, $B =$ constant vector.

If $|A| \neq 0$: unique solution $X = A^{-1}B$

Setup Example

System: $3x - 2y = 8, \quad 5x + 4y = 6$

$$\underbrace{\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 8 \\ 6 \end{bmatrix}}_{B}$$

Steps

1. Write A, X, B

Solved: $3x - 2y = 8, \ 5x + 4y = 6$ (GTU W'23)

Full Solution

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}; \quad |A| = 12 + 10 = 22$$

$$\text{adj}(A) = \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}; \quad A^{-1} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 32 + 12 \\ -40 + 18 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 44 \\ -22 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boxed{x = 2, \quad y = -1}$$

Solved: $5x + 2y = 4,\ 7x + 3y = 5$ (GTU W'22)

Full Solution

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}; \quad |A| = 15 - 14 = 1$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\boxed{x = 2, \quad y = -3}$$

Solved: $2x + 3y = 7,\ 4x - y = 9$ (GTU Su'23)

Full Solution

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}; \quad |A| = -2 - 12 = -14$$

$$\text{adj}(A) = \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}; \quad A^{-1} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$$

$$X = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -7 - 27 \\ -28 + 18 \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -34 \\ -10 \end{bmatrix} = \begin{bmatrix} \frac{17}{7} \\ \frac{5}{7} \end{bmatrix}$$

$$\boxed{x = \frac{17}{7}, \quad y = \frac{5}{7}}$$

Solved: $3x + y = 9,\ 2x - 3y = -5$ (GTU Su'25/N-W'25)

Full Solution

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}; \quad |A| = -9 - 2 = -11$$

$$\text{adj}(A) = \begin{bmatrix} -3 & -1 \\ -2 & 3 \end{bmatrix}; \quad A^{-1} = \frac{1}{-11} \begin{bmatrix} -3 & -1 \\ -2 & 3 \end{bmatrix}$$

$$X = \frac{1}{-11} \begin{bmatrix} -3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ -5 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -27 + 5 \\ -18 - 15 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -22 \\ -33 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\boxed{x = 2, \quad y = 3}$$

1.12 GTU PYQ — Practice Session

MCQ Bank — Orders and Elements

Quick-fire MCQs (1 mark each)

1. Order of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = 2 \times 3$
2. Order of $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 3 \times 1$
3. Order of $[5\ 6\ 7\ 8] = 1 \times 4$
4. If $\begin{bmatrix} x-3 & 0 \\ 0 & y+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ then $x = \mathbf{8}$, $y = \mathbf{3}$
5. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
6. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $4A = \begin{bmatrix} 8 & 4 \\ 12 & 16 \end{bmatrix}$
7. If A is 2×3 and B is 3×4, order of $AB = 2 \times 4$
8. $(A^{-1})^{-1} = A \quad ; \quad (AB)^{-1} = B^{-1}A^{-1}$

MCQ: $2A - 3I$ and A^2 Calculations (GTU W'22, W'23)

W'22: Find $2A - 3I$ for $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$2A = \begin{bmatrix} 4 & 2 \\ 0 & 6 \end{bmatrix}$; $3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$2A - 3I = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

W'23: Find A^2 for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

3-mark: $A + A^T + I$ (GTU Su'24)

Find $A + A^T + I$ for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A + A^T + I = \begin{bmatrix} 1+1+1 & -1+2+0 \\ 2-1+0 & 3+3+1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}}$$

3-mark: Matrix Method $3x - y = 1,\ 2x + y = 4$ (GTU W'24)

Solve by Matrix Method

$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}; \quad |A| = 3 + 2 = 5$

$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

$X = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{x = 1, \quad y = 2}$

3-mark: Matrix Mult Proof with $x^3 = 1$ (GTU W'22)

If ω is cube root of unity ($\omega^3 = 1,\ 1 + \omega + \omega^2 = 0$)

Show $\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}^2 = [\dots]$ (rows are cyclic permutations)

The product of any two rows/columns uses $\omega^3 = 1$:
Row 1 $\cdot$ Row 1: $1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega = 0$
Row 1 $\cdot$ Col 1: $1 + \omega^2 + \omega^4 = 0$
Hence $A^2 = O$ (zero matrix) ✓

4-mark: Find AB given $A + B$ and $A - B$ (GTU Su'24)

Given: $A + B = \begin{bmatrix} 6 & 6 \\ 0 & 8 \end{bmatrix}$, $A - B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$

Step 1: Find A and B

$$2A = (A + B) + (A - B) = \begin{bmatrix} 8 & 6 \\ 4 & 10 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$2B = (A + B) - (A - B) = \begin{bmatrix} 4 & 6 \\ -4 & 6 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}$$

Step 2: Find AB

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 - 6 & 12 + 9 \\ 4 - 10 & 6 + 15 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 21 \\ -6 & 21 \end{bmatrix}}$$

4-mark: Prove $(AB)^{-1} = B^{-1}A^{-1}$ (GTU W'24)

Proof

To show: $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$

LHS:

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$ ✓

RHS:

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I$ ✓

Since both products equal I, by definition $(AB)^{-1} = B^{-1}A^{-1}$. $\square$

4-mark: A^{-1} for 3×3 (GTU W'24)

Find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

$|A|$**:** $= 1(-2-2) - 2(3-4) + 3(6+8) = 1(-4) - 2(-1) + 3(14) = -4 + 2 + 42 = 40$

Cofactors:

$C_{11} = -4,\ C_{12} = 1,\ C_{13} = 14$
$C_{21} = (-1)^{2+1}(2-6) = 4,\ C_{22} = (-1)^{2+2}(1-12) = -11,\ C_{23} = (-1)^{2+3}(2-8) = 6$
$C_{31} = (2+6) = 8,\ C_{32} = (-1)^{3+2}(1-9) = 8,\ C_{33} = (-2-6) = -8$

$\text{adj}\, A = \begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$

$A^{-1} = \frac{1}{40} \begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$

4-mark: Solve $2x + 3y = 1,\ \ y - 4x = 2$ (GTU Su'24)

Full Solution

Rewrite: $2x + 3y = 1, \quad -4x + y = 2$

$A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}; \quad |A| = 2 + 12 = 14$

$A^{-1} = \dfrac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

$X = \dfrac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \dfrac{1}{14} \begin{bmatrix} 1 - 6 \\ 4 + 4 \end{bmatrix} = \dfrac{1}{14} \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\boxed{x = -\frac{5}{14}, \quad y = \frac{4}{7}}$

Unit 1 — Summary and Key Formulae

Must-Know Formulae (Exam Cheat-sheet)

Det 2×2:	$\lvert A\rvert = ad - bc$
Adj 2×2:	Swap diag, negate off-diag
Inverse:	$A^{-1} = \frac{\text{adj}\,A}{\lvert A\rvert}$
Char. eq. 2×2:	$\lambda^2 - (\text{tr}A)\lambda + \lvert A\rvert = 0$
Cayley-Hamilton:	Replace $\lambda \to A$; holds for A
Matrix system:	$X = A^{-1}B$ if $\lvert A\rvert \neq 0$
Transpose:	$(AB)^T = B^T A^T$
Inverse:	$(AB)^{-1} = B^{-1}A^{-1}$

Unit 1 PYQ Frequency

4-mark: Inverse (3×3) & Linear systems — *appear in every paper*
3-mark: Cayley-Hamilton proofs — *W'24, Su'24 both had this*
MCQ: Order, adj, $(AB)^{-1} = B^{-1}A^{-1}$ — *very frequent*

Unit 1 Complete!

Next: Unit 2 — Successive Differentiation