Unit 1: Matrices

Applied Mathematics — 4320001 / DI02000011

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Syllabus Coverage

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1.1 Introduction to Matrices

Definition · Notation · Order

Why Do We Need Matrices?

Real-World Motivation

- ➥ In ECE: nodal voltages in a circuit with n nodes give n simultaneous equations.
- ➥ Solving them by hand (substitution) is impractical for $n > 3$.
- ➥ Matrices provide a **compact, systematic** way to write and solve such systems.
- ➥ Also used in: image processing, network analysis, machine learning, 3D graphics.

Everyday Example: Spreadsheet

A **spreadsheet** is a matrix! Each cell (i, j) holds a value.
Operations on spreadsheet columns/rows are exactly matrix operations.

Definition of a Matrix

Definition

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**, enclosed in square brackets $[\,\cdot\,]$ or parentheses $(\,\cdot\,)$.

General form of $A_{m\times n}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Key Terms

m = number of **rows**
n = number of **columns**
a_{ij} = element at row i, column j
Order $= m \times n$
Also written: $A = [a_{ij}]_{m\times n}$

Order of a Matrix — Examples

Order $=$ rows $\times$ columns

Matrix	Rows	Order
$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$	2 rows, 2 cols	2×2
$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \end{bmatrix}$	2 rows, 3 cols	2×3
$\begin{bmatrix} 0 & -4 & 2 \end{bmatrix}$	1 row, 3 cols	1×3 (Row matrix)
$\begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}$	3 rows, 1 col	3×1 (Column matrix)

GTU MCQ Trick

Always count **rows first**, then columns. Order $= m \times n$ NOT $n \times m$.

Identifying Elements by Position

Element a_{ij}

a_{ij} means the element in the ***i*th row** and ***j*th column**.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{Order: } 3 \times 3$$

$a_{11} = 1$ (row 1, col 1)
$a_{12} = 2$ (row 1, col 2)
$a_{23} = 6$ (row 2, col 3)
$a_{32} = 8$ (row 3, col 2)
$a_{33} = 9$ (row 3, col 3)

GTU Q: If $x - 3 = 5$ (equality condition)

1.2 Types of Matrices

Types of Matrices — Part 1

Shape-Based Types

Type	Definition
Row Matrix	Exactly **1 row**: order $1 \times n$. E.g. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
Column Matrix	Exactly **1 column**: order $m \times 1$. E.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
Square Matrix	$m = n$ (equal rows and columns). E.g. 2×2, 3×3
Rectangular Matrix	$m \neq n$. E.g. 2×3, 3×4
Zero/Null Matrix	All elements $= 0$. Denoted O or $[0]$

Examples

Row: $\begin{bmatrix} 3 & -1 & 7 \end{bmatrix}$ Column: $\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$ Square: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Zero: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Types of Matrices — Part 2 (Square Matrices)

Special Square Matrix Types

Type	Definition
Identity Matrix I	Diagonal $= 1$, all others $= 0$. $AI = IA = A$
Diagonal Matrix	All non-diagonal elements $= 0$
Scalar Matrix	Diagonal matrix where all diagonal $=$ same value k
Upper Triangular	All elements *below* main diagonal $= 0$
Lower Triangular	All elements *above* main diagonal $= 0$
Symmetric Matrix	$A^T = A$; $a_{ij} = a_{ji}$
Skew-Symmetric	$A^T = -A$; diagonal elements $= 0$

Identity and Diagonal Examples

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Diag: $\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$ UT: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Types of Matrices — Symmetric & Skew-Symmetric

Symmetric: $A^T = A$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Note: $a_{ij} = a_{ji}$ for all i, j.

Property: $A + A^T$ is always symmetric.

Skew-Sym: $A^T = -A$

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Note: $a_{ii} = 0$ always.

Property: $A - A^T$ is always skew-symmetric.

Important Identity

Every square matrix A can be written as: $A = \underbrace{\frac{A + A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{Skew-Symmetric}}$

1.3 Matrix Operations

Addition · Subtraction · Scalar Multiplication

Equality of Matrices

Condition for Equality

Two matrices A and B are **equal** ($A = B$) if and only if:
(i) They have the **same order**, AND
(ii) Every **corresponding element** is equal: $a_{ij} = b_{ij}$ for all i, j.

GTU Example (W'22)

If $\begin{bmatrix} x-3 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$, find x.

Comparing $(1, 1)$ elements: $x - 3 = 5 \Rightarrow \boxed{x = 8}$

Note

Matrices of different orders **cannot** be equal.

Matrix Addition and Subtraction

Rule

Add/subtract matrices of the **same order** by adding/subtracting corresponding elements.

$$C = A \pm B \;\Rightarrow\; c_{ij} = a_{ij} \pm b_{ij}$$

Example: $A + B$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

GTU: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $2A - 3I$

Scalar Multiplication

Rule

Multiply every element of the matrix by the scalar k:

$$kA = [k \cdot a_{ij}]$$

Example: $4A$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, then $4A = \begin{bmatrix} 4 \times 1 & 4 \times 2 \\ 4 \times 3 & 4 \times (-1) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & -4 \end{bmatrix}$

(GTU W'23: verified)

Properties of Scalar Multiplication

$k(A+B) = kA + kB \quad (k_1 + k_2)A = k_1 A + k_2 A \quad k_1(k_2 A) = (k_1 k_2)A$

$1 \cdot A = A$

1.4 Matrix Multiplication

Condition for Matrix Multiplication

Condition

$A \times B$ is defined only if the number of **columns of** A equals the number of **rows of** B.

Order Rule

If A is $m \times p$ and B is $p \times n$, then AB is $m \times n$.

$$A_{m\times \mathbf{p}} \qquad B_{\mathbf{p}\times n} \qquad = \qquad C_{m\times n}$$

must match

GTU MCQ (W'22)

If A is 2×3 and B is 3×4, the order of $AB = \mathbf{2} \times \mathbf{4}$.

How to Multiply Matrices

Formula

$C = AB$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$ (dot product of ith row of A with jth column of B).

Example: Row × Column (GTU W'24)

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} = 1(4) + 2(5) + 3(-1) = 4 + 10 - 3 = \mathbf{11}$$

Example: 2 × 2 Multiplication

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Matrix Multiplication — 3×3 Example

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find A^2 (GTU W'23)

$$A^2 = A \times A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1) + 1(1) & 1(1) + 1(1) \\ 1(1) + 1(1) & 1(1) + 1(1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Properties of Matrix Multiplication

Associative:	$(AB)C = A(BC)$
Distributive:	$A(B + C) = AB + AC$
Identity:	$AI = IA = A$
NOT Commutative:	$AB \neq BA$ (in general!)
Zero:	$AO = OA = O$

1.5 Transpose of a Matrix

Transpose: Definition and Examples

Definition

The **transpose** of matrix A (denoted A^T or A') is obtained by **interchanging rows and columns**. If $A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$.

2 × 2 Example (GTU W'24)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2 × 3 Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

Properties of Transpose

Key Properties (Exam Important)

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T \quad$ *(order reverses!)*
5. $(ABC)^T = C^T B^T A^T$

GTU Q (Su'24): Find $A + A^T + I$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A + A^T + I = \begin{bmatrix} 1+1+1 & -1+2+0 \\ 2-1+0 & 3+3+1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}$$

1.6 Determinant of a Matrix

Determinant of a 2 × 2 Matrix

Definition

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\quad |A| = \det(A) = ad - bc$

Examples

$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 2(4) - 3(1) = 8 - 3 = \mathbf{5}$ $\quad \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} = (-1)(-4) - (2)(3) = 4$

Singular Matrix Condition

A matrix A is **singular** if $|A| = 0$ (no inverse exists).

GTU Example: Find x if $\begin{bmatrix} x & 2 \\ 3 & 1 \end{bmatrix}$ is singular.

$|A| = x - 6 = 0 \Rightarrow \boxed{x = 6}$

Minors and Cofactors (for 3×3)

Definitions

Minor M_{ij}: determinant of the matrix obtained by deleting row i and column j.

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

Sign Pattern of $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

(top-left is always $+$; signs alternate like a chessboard)

Example for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

Determinant of a 3×3 Matrix (Expansion)

Expansion along Row 1

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Full Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

$$= 1(0 - 24) - 2(0 - 20) + 3(0 - 5)$$

$$= -24 + 40 - 15 = \mathbf{1}$$

Properties of Determinants

✓ $|AB| = |A||B|$ ✓ $|A^T| = |A|$ ✓ $|kA| = k^n|A|$ (for $n \times n$) ✓ If two rows are identical $\Rightarrow |A| = 0$

1.7 Singular & Non-Singular Matrices

Singular vs Non-Singular

Singular Matrix

$|A| = 0$

No inverse exists.
System $AX = B$ may have no solution or infinitely many.

Non-Singular Matrix

$|A| \neq 0$

Inverse exists.
System $AX = B$ has a unique solution.

GTU W'22: Find x if singular

$$A = \begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}, \ |A| = 0$$

Expanding: $(3-x)\big[(4-x)(-1-x)+4\big]-2\big[(-1-x)+2\big]+2\big[-4+2(4-x)\big]=0$
Solving gives $x = 0,\ 1,\ 5$ (roots of the characteristic-like condition).

Checking Singularity — Quick Examples

Non-singular: has inverse

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad |A| = 4 - 6 = -2 \neq 0 \checkmark$$

Singular: no inverse

$$B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \quad |B| = 4 - 4 = 0 ✗$$

Row 2 is exactly half of Row 1 — linearly dependent!

Key Insight

A matrix is singular $\Leftrightarrow$ its rows (or columns) are **linearly dependent**.

1.8 Adjoint (Adjugate) of a Matrix

Adjoint: Definition

Definition

The **adjoint** (or adjugate) of A is the **transpose of the cofactor matrix**:

$$\text{adj}(A) = [C_{ij}]^T$$

where $C_{ij} = (-1)^{i+j} M_{ij}$ are the cofactors.

2×2 Shortcut (Very Useful!)

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\quad \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Swap main diagonal; negate off-diagonal.

Adjoint Examples (GTU MCQs)

Su'24: adj of $\begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$

Use shortcut: swap diagonal, negate off-diagonal

$\text{adj} = \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}$ **Answer: B**

W'24: adj of $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$\text{adj} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ **Answer: A**

W'23: adj of $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

$\text{adj} = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ **Answer: D**

Adjoint of a 3×3 Matrix

Step-by-step: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ — wait, use 3×3:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Step 1: Cofactor Matrix

$$C_{11} = +\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} = -24; \;\; C_{12} = -\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} = +20; \;\; C_{13} = +\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} = -5$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} = +18; \;\; C_{22} = +\begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = -15; \;\; C_{23} = -\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = +4$$

$$C_{31} = +\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5; \;\; C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4; \;\; C_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

Step 2: Transpose the cofactor matrix

$$\text{adj}(A) = [C_{ij}]^T = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

1.9 Inverse of a Matrix

Inverse: Definition and Formula

Definition

If A is a non-singular square matrix, then A^{-1} (inverse) satisfies:

$$AA^{-1} = A^{-1}A = I$$

Formula

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}}$$

Condition: $|A| \neq 0$ (non-singular).

GTU MCQ (Su'24, W'22)

✓ $(A^{-1})^{-1} = A$ ✓ If $AB = I$ then $B = A^{-1}$ ✓ $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of a 2×2 Matrix (GTU W'23)

Find adj and inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Step 1: $|A| = (1)(4) - (2)(3) = 4 - 6 = -2 \neq 0$ ✓

Step 2: $\text{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Step 3: $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Verify: $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Inverse of a 3×3 Matrix — Full Method (GTU W'23)

Find A^{-1} for $A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 3 & -2 & 5 \end{bmatrix}$

Step 1: $|A|$
$|A| = 1(0 \cdot 5 - 6 \cdot (-2)) - (-1)(4 \cdot 5 - 6 \cdot 3) + 2(4 \cdot (-2) - 0 \cdot 3)$
$= 1(12) + 1(2) + 2(-8) = 12 + 2 - 16 = -2$

Step 2: Cofactors
$C_{11} = 12,\ C_{12} = -2,\ C_{13} = -8$
$C_{21} = 1,\ C_{22} = -1,\ C_{23} = -1$
$C_{31} = -6,\ C_{32} = 2,\ C_{33} = 4$

Step 3: **adj** $= [C_{ij}]^T = \begin{bmatrix} 12 & 1 & -6 \\ -2 & -1 & 2 \\ -8 & -1 & 4 \end{bmatrix}$

Step 4: $A^{-1} = \dfrac{1}{-2} \begin{bmatrix} 12 & 1 & -6 \\ -2 & -1 & 2 \\ -8 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -\frac{1}{2} & 3 \\ 1 & \frac{1}{2} & -1 \\ 4 & \frac{1}{2} & -2 \end{bmatrix}$

Inverse — Another 3×3 (GTU Su'24)

Find A^{-1} for $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$

Step 1: $|A| = 3(1-0) + 1(4+5) + 2(0-5) = 3 + 9 - 10 = \mathbf{2}$

Step 2: Cofactors

$C_{11} = +\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1;\ \ C_{12} = -\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} = -(4+5) = -9;\ \ C_{13} = +\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} = -5$

$C_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1;\ \ C_{22} = +\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = -7;\ \ C_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & 0 \end{vmatrix} = 5$

$C_{31} = +\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1;\ \ C_{32} = -\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = 11;\ \ C_{33} = +\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = 7$

adj$= \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & 5 & 7 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & 5 & 7 \end{bmatrix}$

1.10 Cayley-Hamilton Theorem

Statement and Characteristic Equation

Cayley-Hamilton Theorem

Every square matrix satisfies its own **characteristic equation**.

Characteristic Equation

The characteristic equation of A is $\det(A - \lambda I) = 0$.

For 2×2**:** $\lambda^2 - (\operatorname{tr} A)\lambda + |A| = 0$

where $\operatorname{tr} A = a_{11} + a_{22}$ (sum of diagonal).

For 3×3**:** $\lambda^3 - (\operatorname{tr} A)\lambda^2 + (\text{sum of } 2 \times 2 \text{ principal minors})\lambda - |A| = 0$

What it means

Replace λ with A (and scalar 0 with zero matrix O) — the equation still holds!

Cayley-Hamilton: 2×2 Example (GTU Su'24)

Prove $A^2 - 4A + 7I = 0$ for $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

Step 1: Characteristic equation

$\text{tr}(A) = 2 + 2 = 4; \quad |A| = 4 + 3 = 7$

Char. eq.: $\lambda^2 - 4\lambda + 7 = 0$ ✓ (matches given)

Step 2: Verify by direct computation

$$A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$$

$$A^2 - 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 8 + 7 & 12 - 12 + 0 \\ -4 + 4 + 0 & 1 - 8 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

Cayley-Hamilton: 2×2 Example (GTU W'24)

Prove $A^2 - 7A + 14I = 0$ for $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

Characteristic equation:

$\mathrm{tr}(A) = 3 + 4 = 7; \quad |A| = 12 + 2 = 14$

Char. eq.: $\lambda^2 - 7\lambda + 14 = 0$ ✓

Verify:

$$A^2 = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 7 & 14 \\ -7 & 14 \end{bmatrix}$$

$$A^2 - 7A + 14I = \begin{bmatrix} 7 & 14 \\ -7 & 14 \end{bmatrix} - \begin{bmatrix} 21 & 14 \\ -7 & 28 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

Cayley-Hamilton: Finding A^{-1} Using the Theorem

Trick: Use the characteristic equation to find A^{-1}

If $A^2 - 4A + 7I = 0$, then:

$$A^2 - 4A + 7I = 0$$
$$\Rightarrow A^2 - 4A = -7I$$
$$\Rightarrow A(A - 4I) = -7I$$
$$\Rightarrow A \cdot \frac{-(A - 4I)}{7} = I$$
$$\therefore A^{-1} = \frac{4I - A}{7} = \frac{1}{7}\begin{bmatrix} 4-2 & 0-3 \\ 0+1 & 4-2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

Exam Tip

When asked to find A^{-1} using Cayley-Hamilton, use this approach — much faster than cofactor method for 2×2.

Cayley-Hamilton: 3×3 Example (GTU Su'23)

Prove $A^2 - 4A - 5I = 0$ for $A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

Char. eq.: $\text{tr}(A) = 6$; $|A| = 2(4-2) - 2(2-2) + 2(1-2) = 4 - 0 - 2 = 2$...
Actually verify by computing $A^2 - 4A - 5I$ directly.

$$A^2 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 10 & 10 & 12 \\ 6 & 9 & 10 \\ 5 & 7 & 8 \end{bmatrix}$$

$$4A = \begin{bmatrix} 8 & 8 & 8 \\ 4 & 8 & 8 \\ 4 & 4 & 8 \end{bmatrix}; \quad 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 10-8-5 & 10-8-0 & 12-8-0 \\ 6-4-0 & 9-8-5 & 10-8-0 \\ 5-4-0 & 7-4-0 & 8-8-5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 4 \\ 2 & -4 & 2 \\ 1 & 3 & -5 \end{bmatrix}$$ ... *(Note: GTU gives this as a proof; verify in exam.)*

Cayley-Hamilton: 3×3 (GTU W'24)

Prove $A^3 - 4A^2 - 3A + 11I = 0$ for $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

Step 1: $|A| = 1(0+2) - 3(6+1) + 2(4-0) = 2 - 21 + 8 = -11$

Step 2: $\mathrm{tr}(A) = 1+0+3 = 4$; Sum of 2×2 principal minors $= \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix}$
$= (0-6) + (3-2) + (0+2) = -3$

Char. eq.: $\lambda^3 - 4\lambda^2 - 3\lambda + 11$... wait, signs: $\lambda^3 - 4\lambda^2 + (-3)\lambda - (-11)$
$= \lambda^3 - 4\lambda^2 - 3\lambda + 11 = 0$ ✓

By Cayley-Hamilton: $A^3 - 4A^2 - 3A + 11I = 0$ ✓

1.11 System of Linear Equations

Matrix Method: $AX = B$

Method

Write the system as $AX = B$ where $A =$ coefficient matrix, $X =$ variable vector, $B =$ constant vector.

If $|A| \neq 0$: unique solution $X = A^{-1}B$

Setup Example

System: $3x - 2y = 8, \quad 5x + 4y = 6$

$$\underbrace{\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 8 \\ 6 \end{bmatrix}}_{B}$$

Steps

1. Write A, X, B

Solved: $3x - 2y = 8,\ 5x + 4y = 6$ (GTU W'23)

Full Solution

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}; \quad |A| = 12 + 10 = 22$$

$$\text{adj}(A) = \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}; \quad A^{-1} = \frac{1}{22}\begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{22}\begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 32 + 12 \\ -40 + 18 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 44 \\ -22 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boxed{x = 2, \quad y = -1}$$

Solved: $5x + 2y = 4,\ 7x + 3y = 5$ (GTU W'22)

Full Solution

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}; \quad |A| = 15 - 14 = 1$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\boxed{x = 2, \quad y = -3}$$

Solved: $2x + 3y = 7,\ 4x - y = 9$ (GTU Su'23)

Full Solution

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}; \quad |A| = -2 - 12 = -14$$

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}; \quad A^{-1} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$$

$$X = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -7 - 27 \\ -28 + 18 \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -34 \\ -10 \end{bmatrix} = \begin{bmatrix} \frac{17}{7} \\ \frac{5}{7} \end{bmatrix}$$

$$\boxed{x = \frac{17}{7}, \quad y = \frac{5}{7}}$$

Solved: $3x + y = 9,\ 2x - 3y = -5$ (GTU Su'25/N-W'25)

Full Solution

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}; \quad |A| = -9 - 2 = -11$$

$$\text{adj}(A) = \begin{bmatrix} -3 & -1 \\ -2 & 3 \end{bmatrix}; \quad A^{-1} = \frac{1}{-11}\begin{bmatrix} -3 & -1 \\ -2 & 3 \end{bmatrix}$$

$$X = \frac{1}{-11}\begin{bmatrix} -3 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ -5 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -27 + 5 \\ -18 - 15 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -22 \\ -33 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\boxed{x = 2, \quad y = 3}$$

1.12 GTU PYQ — Practice Session

MCQ Bank — Orders and Elements

Quick-fire MCQs (1 mark each)

1. Order of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = 2 \times 3$
2. Order of $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 3 \times 1$
3. Order of $[5\ 6\ 7\ 8] = 1 \times 4$
4. If $\begin{bmatrix} x-3 & 0 \\ 0 & y+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ then $x = \mathbf{8}$, $y = \mathbf{3}$
5. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
6. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $4A = \begin{bmatrix} 8 & 4 \\ 12 & 16 \end{bmatrix}$
7. If A is 2×3 and B is 3×4, order of $AB = 2 \times 4$
8. $(A^{-1})^{-1} = A \quad ; \quad (AB)^{-1} = B^{-1}A^{-1}$

MCQ: $2A - 3I$ and A^2 Calculations (GTU W'22, W'23)

W'22: Find $2A - 3I$ for $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$2A = \begin{bmatrix} 4 & 2 \\ 0 & 6 \end{bmatrix}; \quad 3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$2A - 3I = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

W'23: Find A^2 for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

3-mark: $A + A^T + I$ (GTU Su'24)

Find $A + A^T + I$ for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A + A^T + I = \begin{bmatrix} 1+1+1 & -1+2+0 \\ 2-1+0 & 3+3+1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}}$$

3-mark: Matrix Method $3x - y = 1,\ 2x + y = 4$ (GTU W'24)

Solve by Matrix Method

$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}; \quad |A| = 3 + 2 = 5$

$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

$X = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{x = 1, \quad y = 2}$

3-mark: Matrix Mult Proof with $x^3 = 1$ (GTU W'22)

If ω is cube root of unity ($\omega^3 = 1,\ 1 + \omega + \omega^2 = 0$)

Show $\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}^2 = [\ldots]$ (rows are cyclic permutations)

The product of any two rows/columns uses $\omega^3 = 1$:
Row 1 $\cdot$ Row 1: $1 + \omega^2 + \omega^4 = 1 + \omega^2 + \omega = 0$
Row 1 $\cdot$ Col 1: $1 + \omega^2 + \omega^4 = 0$
Hence $A^2 = O$ (zero matrix) ✓

4-mark: Find AB given $A+B$ and $A-B$ (GTU Su'24)

Given: $A+B=\begin{bmatrix}6 & 6\\0 & 8\end{bmatrix}$, $A-B=\begin{bmatrix}2 & 0\\4 & 2\end{bmatrix}$

Step 1: Find A and B

$$2A=(A+B)+(A-B)=\begin{bmatrix}8 & 6\\4 & 10\end{bmatrix}\Rightarrow A=\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}$$

$$2B=(A+B)-(A-B)=\begin{bmatrix}4 & 6\\-4 & 6\end{bmatrix}\Rightarrow B=\begin{bmatrix}2 & 3\\-2 & 3\end{bmatrix}$$

Step 2: Find AB

$$AB=\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}\begin{bmatrix}2 & 3\\-2 & 3\end{bmatrix}=\begin{bmatrix}8-6 & 12+9\\4-10 & 6+15\end{bmatrix}=\boxed{\begin{bmatrix}2 & 21\\-6 & 21\end{bmatrix}}$$

4-mark: Prove $(AB)^{-1} = B^{-1}A^{-1}$ (GTU W'24)

Proof

To show: $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$

LHS:
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$ ✓

RHS:
$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I$ ✓

Since both products equal I, by definition $(AB)^{-1} = B^{-1}A^{-1}$. □

4-mark: A^{-1} for 3×3 (GTU W'24)

Find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

$|A|$: $= 1(-2-2) - 2(3-4) + 3(6+8) = 1(-4) - 2(-1) + 3(14) = -4 + 2 + 42 = 40$

Cofactors:

$C_{11} = -4,\ C_{12} = 1,\ C_{13} = 14$

$C_{21} = (-1)^{2+1}(2-6) = 4,\ C_{22} = (-1)^{2+2}(1-12) = -11,\ C_{23} = (-1)^{2+3}(2-8) = 6$

$C_{31} = (2+6) = 8,\ C_{32} = (-1)^{3+2}(1-9) = 8,\ C_{33} = (-2-6) = -8$

$$\text{adj } A = \begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

4-mark: Solve $2x + 3y = 1,\ y - 4x = 2$ (GTU Su'24)

Full Solution

Rewrite: $2x + 3y = 1, \quad -4x + y = 2$

$A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}; \quad |A| = 2 + 12 = 14$

$A^{-1} = \frac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

$X = \frac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 - 6 \\ 4 + 4 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\boxed{x = -\frac{5}{14}, \quad y = \frac{4}{7}}$

Unit 1 — Summary and Key Formulae

Must-Know Formulae (Exam Cheat-sheet)

Det 2×2**:**	$\lvert A\rvert = ad - bc$
Adj 2×2**:**	Swap diag, negate off-diag
Inverse:	$A^{-1} = \frac{\text{adj}\,A}{\lvert A\rvert}$
Char. eq. 2×2**:**	$\lambda^2 - (\text{tr}A)\lambda + \lvert A\rvert = 0$
Cayley-Hamilton:	Replace $\lambda \to A$; holds for A
Matrix system:	$X = A^{-1}B$ if $\lvert A\rvert \neq 0$
Transpose:	$(AB)^T = B^T A^T$
Inverse:	$(AB)^{-1} = B^{-1}A^{-1}$

Unit 1 PYQ Frequency

4-mark: Inverse (3×3) & Linear systems — *appear in every paper*
3-mark: Cayley-Hamilton proofs — *W'24, Su'24 both had this*
MCQ: Order, adj, $(AB)^{-1} = B^{-1}A^{-1}$ — *very frequent*

Unit 1 Complete!

Next: Unit 2 — Successive Differentiation

Unit 2: Differentiation and its Applications

Applied Mathematics — 4320001 / DI02000011

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April 2, 2026

Unit 2 — Table of Contents

Syllabus Coverage

Weightage: ≈16 marks (23%) | Hours: 10–11

2.1 Limits and Continuity

Concept of a Limit

Definition

$\lim_{x \to a} f(x) = L$ means: as x gets arbitrarily close to a, $f(x)$ gets arbitrarily close to L.

Key Limit Rules

Sum:	$\lim[f+g] = \lim f + \lim g$
Product:	$\lim[f \cdot g] = \lim f \cdot \lim g$
Quotient:	$\lim \dfrac{f}{g} = \dfrac{\lim f}{\lim g}, \quad \lim g \neq 0$
Scalar:	$\lim[kf] = k \lim f$

Standard Limits (Memorise)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad \lim_{x \to 0} (1+x)^{1/x} = e$$

Continuity

Condition for Continuity at $x = a$

A function $f(x)$ is continuous at $x = a$ if all three hold:

1. $f(a)$ is defined
2. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ (left limit = right limit)
3. $\lim_{x \to a} f(x) = f(a)$

Intuition

A function is continuous if you can draw its graph **without lifting the pen**.

Differentiability $\Rightarrow$ **Continuity** (but not vice versa).

E.g. $f(x) = |x|$ is continuous but *not differentiable* at $x = 0$.

2.2 Derivative from First Principles

Definition of Derivative

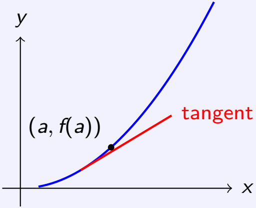
First Principles (ab-initio definition)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Also written as $\dfrac{dy}{dx}$, y', or Df.

Geometric Meaning

$f'(a) =$ **slope of tangent** to the curve $y = f(x)$ at $x = a$.



First Principles: Derivative of $\sqrt{x}$ (GTU W'22)

Find $\frac{d}{dx}(\sqrt{x})$ from first principles

$f(x) = \sqrt{x}, \quad f(x+h) = \sqrt{x+h}$

$f'(x) = \lim_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

Rationalise: $= \lim_{h\to 0} \dfrac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h\to 0} \dfrac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \dfrac{h}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h\to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

$$\boxed{\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}}$$

First Principles: Derivative of e^x (GTU Su'24, Su'25)

Find $\frac{d}{dx}(e^x)$ from first principles

$f(x) = e^x, \quad f(x+h) = e^{x+h} = e^x \cdot e^h$

$f'(x) = \lim_{h\to 0} \dfrac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \dfrac{e^x(e^h - 1)}{h}$

$= e^x \lim_{h\to 0} \dfrac{e^h - 1}{h}$

Using standard limit $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$:

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

First Principles: Derivative of $\cos x$ (GTU Su'23)

Find $\frac{d}{dx}(\cos x)$ from first principles

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

Use: $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$= \lim_{h\to 0} \frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot \lim_{h\to 0}\frac{\sin(h/2)}{h/2}$$

$$= -\sin x \cdot 1 = \boxed{-\sin x}$$

2.3 Standard Derivatives

Standard Derivatives Table (Part 1)

Algebraic and Exponential

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

GTU MCQ Examples

- ✓ $\frac{d}{dx}(x^3 + 3^x + 3^3) = 3x^2 + 3^x \ln 3$ (W'22)
- ✓ $\frac{d}{dx}(x^3 + e^x + 2) = 3x^2 + e^x$ (N-W'25)
- ✓ $\frac{d}{dx}(x^3 + 1) = 3x^2$ (W'24)

Standard Derivatives Table (Part 2)

Trigonometric

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Inverse Trigonometric

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

GTU MCQ: Tricky Standard Derivative (W'24)

$$\frac{d}{dx}(\sec^2 x - \tan^2 x) = ?$$

Key insight: $\sec^2 x - \tan^2 x = 1$ (identity!)

$$\therefore \frac{d}{dx}(\sec^2 x - \tan^2 x) = \frac{d}{dx}(1) = \boxed{0}$$

Don't differentiate term-by-term blindly — use identities first!

GTU Su'24: $\frac{d}{dx}(\tan^{-1} x + \cot^{-1} x)$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \text{ (constant!)}$$

$\therefore$ derivative $= \boxed{0}$

2.4 Rules: Chain, Product & Quotient

Chain Rule

Chain Rule

If $y = f(g(x))$, then $\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x)$

"Differentiate outside, keep inside, then multiply by derivative of inside."

Examples

$$y = \sin^5 x \qquad \frac{dy}{dx} = 5\sin^4 x \cdot \cos x \qquad \text{(GTU W'23)}$$

$$y = e^{\cos x} \qquad \frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -\sin x\, e^{\cos x}$$

$$y = \log(4x-3) \qquad \frac{dy}{dx} = \frac{1}{4x-3} \cdot 4 = \frac{4}{4x-3} \qquad \text{(GTU N-W'25)}$$

$$y = \tan^{-1}(\sqrt{x}) \qquad \frac{dy}{dx} = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

Product Rule

Product Rule

If $y = u \cdot v$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ ("$uv' + vu'$")

Examples

$y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

$y = e^x \ln x$:

$$\frac{dy}{dx} = e^x \cdot \frac{1}{x} + e^x \ln x = e^x\left(\frac{1}{x} + \ln x\right)$$

Second Derivative using Product Rule (GTU W'22)

$y = e^x \sin x$: $y' = e^x \sin x + e^x \cos x$; $y'' = 2e^x \cos x$.
Can verify: $y'' - 2y' + 2y = 0$.

Quotient Rule

Quotient Rule

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ $\left(\text{“}\dfrac{vu' - uv'}{v^2}\text{”}\right)$

GTU W'24: Differentiate $\dfrac{e^{\cos x}}{\tan x}$

$$u = e^{\cos x}, \quad v = \tan x$$

$$u' = e^{\cos x} \cdot (-\sin x) = -\sin x \cdot e^{\cos x}, \quad v' = \sec^2 x$$

$$\frac{dy}{dx} = \frac{\tan x \cdot (-\sin x \, e^{\cos x}) - e^{\cos x} \cdot \sec^2 x}{\tan^2 x}$$

$$= \frac{e^{\cos x}(-\sin x \tan x - \sec^2 x)}{\tan^2 x}$$

Chain Rule Applied: $y = \log\left(\frac{\sin x}{1+\cos x}\right)$ (GTU W'23)

Find $\frac{dy}{dx}$

$$y = \log(\sin x) - \log(1+\cos x)$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} - \frac{-\sin x}{1+\cos x} = \frac{\cos x}{\sin x} + \frac{\sin x}{1+\cos x}$$

$$= \frac{\cos x(1+\cos x) + \sin^2 x}{\sin x(1+\cos x)} = \frac{\cos x + \cos^2 x + \sin^2 x}{\sin x(1+\cos x)}$$

$$= \frac{\cos x + 1}{\sin x(1+\cos x)} = \frac{1}{\sin x} = \boxed{\csc x}$$

2.5 Implicit Differentiation

Implicit Differentiation: Method

When to use

Use when y is **not** isolated explicitly; e.g. $x^2 + y^2 = r^2$, $\sin(xy) = x + y$.

Steps

1. Differentiate **both sides** with respect to x.
2. Every time you differentiate y, multiply by $\frac{dy}{dx}$ (chain rule).
3. Collect $\frac{dy}{dx}$ terms on one side.
4. Solve for $\frac{dy}{dx}$.

Quick Example: $x^2 + y^2 = a^2$

Implicit: $x + y = \sin(xy)$ (GTU W'22)

Find $\frac{dy}{dx}$

Differentiate both sides w.r.t. x:

$$1 + \frac{dy}{dx} = \cos(xy) \cdot \frac{d}{dx}(xy)$$

$$= \cos(xy) \cdot \left(x\frac{dy}{dx} + y \right)$$

$$1 + \frac{dy}{dx} = x\cos(xy)\frac{dy}{dx} + y\cos(xy)$$

$$\frac{dy}{dx}\left[1 - x\cos(xy)\right] = y\cos(xy) - 1$$

$$\boxed{\frac{dy}{dx} = \frac{y\cos(xy) - 1}{1 - x\cos(xy)}}$$

Implicit: $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (GTU Su'24)

Prove $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

Differentiate both sides w.r.t. x:

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}} \checkmark$$

Implicit: $y = \sin(x + y)$ (GTU W'23)

Find $\frac{dy}{dx}$

Differentiate both sides w.r.t. x:

$$\frac{dy}{dx} = \cos(x + y) \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \cos(x + y) + \cos(x + y)\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[1 - \cos(x + y)\right] = \cos(x + y)$$

$$\boxed{\frac{dy}{dx} = \frac{\cos(x + y)}{1 - \cos(x + y)}}$$

2.6 Parametric Differentiation

When to use

When x and y are both expressed in terms of a third variable (parameter) t.

Formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$$

GTU Su'24: $x = a\cos\theta,\ y = a\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{a\cos\theta}{-a\sin\theta} = \boxed{-\cot\theta}$$

Parametric: $x = \frac{1}{2}(t + \frac{1}{t})$, $y = \frac{1}{2}(t - \frac{1}{t})$ (GTU W'24)

Find $\frac{dy}{dx}$

$$\frac{dx}{dt} = \frac{1}{2}\left(1 - \frac{1}{t^2}\right) = \frac{t^2 - 1}{2t^2}$$

$$\frac{dy}{dt} = \frac{1}{2}\left(1 + \frac{1}{t^2}\right) = \frac{t^2 + 1}{2t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(t^2 + 1)/2t^2}{(t^2 - 1)/2t^2} = \boxed{\frac{t^2 + 1}{t^2 - 1}}$$

2.7 Logarithmic Differentiation

Logarithmic Differentiation: Method

When to use

Use when y has a **variable in the exponent** (e.g. $y = x^x$) or a complicated product/quotient.

Steps

1. Take ln (natural log) of both sides.
2. Simplify using log rules: $\ln(uv) = \ln u + \ln v$; $\ln(u^n) = n \ln u$.
3. Differentiate both sides (implicit).
4. Multiply through by y.

Log Diff: $y = x^x$ (GTU Su'23)

Find $\frac{dy}{dx}$

$\ln y = x \ln x$

Differentiate both sides w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x) = \boxed{x^x(1 + \ln x)}$$

Log Diff: $y = (\tan x)^x$ (GTU N-W'25)

Find $\frac{dy}{dx}$

$$\ln y = x \ln(\tan x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{\sec^2 x}{\tan x} + \ln(\tan x) \cdot 1$$

$$= \frac{x \sec^2 x}{\tan x} + \ln(\tan x)$$

$$\frac{dy}{dx} = y\left[\frac{x \sec^2 x}{\tan x} + \ln(\tan x)\right]$$

$$= \boxed{(\tan x)^x \left[\frac{x \sec^2 x}{\tan x} + \ln(\tan x)\right]}$$

Log Diff: $e^x + e^y = e^{x+y}$ (GTU Su'24)

Find $\frac{dy}{dx}$

$e^x + e^y = e^{x+y} = e^x \cdot e^y$

Divide both sides by $e^x \cdot e^y$:

$e^{-y} + e^{-x} = 1$

Differentiate: $-e^{-y}\dfrac{dy}{dx} - e^{-x} = 0$

$-e^{-y}\dfrac{dy}{dx} = e^{-x}$

$\dfrac{dy}{dx} = -\dfrac{e^{-x}}{e^{-y}} = -e^{y-x} = \boxed{-e^{y-x}}$

2.8 Successive Differentiation

Second and Higher Derivatives

Notation

$$y_1 = y' = \frac{dy}{dx}, \quad y_2 = y'' = \frac{d^2y}{dx^2}, \quad y_n = \frac{d^ny}{dx^n}$$

Standard Second Derivatives

$$\begin{aligned} y &= e^x & y'' &= e^x \\ y &= \sin x & y'' &= -\sin x \\ y &= x^n & y'' &= n(n-1)x^{n-2} \\ y &= \ln x & y'' &= -\frac{1}{x^2} \end{aligned}$$

GTU W'22 MCQ

$$y = e^x + 100x; \quad \frac{d^2y}{dx^2} = e^x + 0 = \boxed{e^x}$$

Prove $y'' - 2y' + 2y = 0$

$y = e^x \sin x$

$y' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

$y'' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$

Verify:

$y'' - 2y' + 2y = 2e^x \cos x - 2e^x(\sin x + \cos x) + 2e^x \sin x$

$= e^x[2\cos x - 2\sin x - 2\cos x + 2\sin x] = e^x[0] = \boxed{0}$ ✓

Successive Diff: $y = 2e^{3x} + 3e^{-2x}$ (GTU W'23, Su'24)

Prove $y'' - y' - 6y = 0$

$y' = 6e^{3x} - 6e^{-2x}$

$y'' = 18e^{3x} + 12e^{-2x}$

Verify:

$y'' - y' - 6y = (18e^{3x} + 12e^{-2x}) - (6e^{3x} - 6e^{-2x}) - 6(2e^{3x} + 3e^{-2x})$

$= (18 - 6 - 12)e^{3x} + (12 + 6 - 18)e^{-2x} = 0 + 0 = \boxed{0}$ ✓

2.9 Maxima and Minima

Maxima and Minima: Method

Test for Critical Points

Step 1: Find $f'(x)$ and solve $f'(x) = 0$ to get critical points $x = a$.

Step 2: Find $f''(a)$:

- $f''(a) < 0 \Rightarrow$ **Local Maximum** at $x = a$
- $f''(a) > 0 \Rightarrow$ **Local Minimum** at $x = a$
- $f''(a) = 0 \Rightarrow$ **Point of inflection** (test further)

GTU W'23 MCQ

If $f(x)$ has a maximum at $x = a$, then $f'(a) = \boxed{0}$.

Max/Min: $f(x) = x^3 - 3x + 11$ (GTU W'23)

Find max and min values

Step 1: $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1) = 0$
$\Rightarrow x = -1$ or $x = 1$

Step 2: $f''(x) = 6x$

At $x = -1$: $f''(-1) = -6 < 0$ ➜ **Local Maximum**
$f(-1) = (-1)^3 - 3(-1) + 11 = -1 + 3 + 11 = \mathbf{13}$

At $x = 1$: $f''(1) = 6 > 0$ ➜ **Local Minimum**
$f(1) = (1)^3 - 3(1) + 11 = 1 - 3 + 11 = \mathbf{9}$

Max value = 13 at $x = -1$; **Min value = 9 at** $x = 1$

Max/Min: $f(x) = x^3 - 4x^2 + 5x + 7$ (GTU W'22, Su'23, Su'25)

Find max and min values

Step 1: $f'(x) = 3x^2 - 8x + 5 = (3x-5)(x-1) = 0$

$\Rightarrow x = 1$ or $x = \dfrac{5}{3}$

Step 2: $f''(x) = 6x - 8$

At $x = 1$: $f''(1) = 6 - 8 = -2 < 0$ ➜ **Local Maximum**

$f(1) = 1 - 4 + 5 + 7 = \mathbf{9}$

At $x = 5/3$: $f''(5/3) = 10 - 8 = 2 > 0$ ➜ **Local Minimum**

$f(5/3) = \dfrac{125}{27} - \dfrac{100}{9} + \dfrac{25}{3} + 7 = \dfrac{125 - 300 + 225 + 189}{27} = \dfrac{239}{27} \approx \mathbf{8.85}$

Max = 9 at $x = 1$; **Min =** $\frac{239}{27}$ **at** $x = \frac{5}{3}$

Max/Min: $f(x) = 2x^3 - 3x^2 - 12x + 5$ (GTU W'24, N-W'25)

Find max and min values

Step 1: $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0$
$\Rightarrow x = 2$ or $x = -1$

Step 2: $f''(x) = 12x - 6$

At $x = -1$: $f''(-1) = -12 - 6 = -18 < 0$ ➜ **Local Maximum**
$f(-1) = -2 - 3 + 12 + 5 = \mathbf{12}$

At $x = 2$: $f''(2) = 24 - 6 = 18 > 0$ ➜ **Local Minimum**
$f(2) = 16 - 12 - 24 + 5 = \mathbf{-15}$

Max = 12 at $x = -1$; **Min =** -15 **at** $x = 2$

2.10 Velocity and Acceleration

Key Relations

If $s = f(t)$ is the displacement at time t:

$$\text{Velocity: } v = \frac{ds}{dt} = s' \qquad \text{Acceleration: } a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = s''$$

Key Exam Points

Particle at rest:	$v = 0$
Uniform velocity:	$a = 0$
Max/min displacement:	$v = 0$
When $v = a$:	Set $s' = s''$ and solve for t

Kinematics: $s = t^3 - 6t^2 + 9t$ (GTU W'22)

Find v and a at $t = 3$; find t when $a = 0$

$s = t^3 - 6t^2 + 9t$

$v = \dfrac{ds}{dt} = 3t^2 - 12t + 9$

$a = \dfrac{d^2s}{dt^2} = 6t - 12$

At $t = 3$:

$v = 3(9) - 12(3) + 9 = 27 - 36 + 9 = \mathbf{0}$

$a = 6(3) - 12 = 18 - 12 = \mathbf{6}\ \text{m/s}^2$

When $a = 0$:

$6t - 12 = 0 \Rightarrow t = \mathbf{2}\ \text{s}$

Kinematics: $s = 2t^3 - 3t^2 - 12t + 7$ (GTU W'23)

Find s and t when $a = 0$

$v = 6t^2 - 6t - 12, \quad a = 12t - 6$

When $a = 0$**:** $12t - 6 = 0 \Rightarrow t = \dfrac{1}{2}$ s

Position at $t = \frac{1}{2}$**:**

$$s = 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - 12\left(\frac{1}{2}\right) + 7$$

$$= \frac{1}{4} - \frac{3}{4} - 6 + 7 = \frac{1-3}{4} + 1 = -\frac{1}{2} + 1 = \boxed{\frac{1}{2}}$$

Kinematics: $s = t^3 + 3t$ — when $v = a$? (GTU Su'24)

Find t when velocity = acceleration

$s = t^3 + 3t$

$v = 3t^2 + 3, \quad a = 6t$

Set $v = a$:

$3t^2 + 3 = 6t$

$3t^2 - 6t + 3 = 0$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0 \Rightarrow \boxed{t = 1}$ s

Kinematics: $S = t^3 - t^2 + 2t + 11$ (GTU W'24)

(a) v at $t = 1$; (b) a at $t = 2$

$$v = \frac{dS}{dt} = 3t^2 - 2t + 2, \quad a = \frac{d^2S}{dt^2} = 6t - 2$$

(a) At $t = 1$: $v = 3 - 2 + 2 = \mathbf{3}$ m/s

(b) At $t = 2$: $a = 12 - 2 = \mathbf{10}$ m/s^2

2.11 GTU PYQ — Practice Session

MCQ Bank (1-mark) — Standard Results

Quick-fire Answers

1. $\dfrac{d}{dx}(x^3 + 3^x + 27) = 3x^2 + 3^x \ln 3$ (W'22)
2. $f(x) = e^{3x} \Rightarrow f'(0) = 3e^0 = \mathbf{3}$ (W'22)
3. $y = e^x + 100x \Rightarrow y'' = e^x$ (W'22)
4. $\dfrac{d}{dx}(\tan x) = \sec^2 x$ (W'23)
5. $\dfrac{d}{dx}(\sin^5 x) = 5 \sin^4 x \cos x$ (W'23)
6. $f'(a) = 0$ at a maximum/minimum (W'23)
7. $\dfrac{d}{dx}(\log x) = 1/x$ (Su'24, W'24)
8. $\dfrac{d}{dx}(\tan^{-1} x + \cot^{-1} x) = 0$ $(=$ const $\pi/2)$ (Su'24)
9. $x = a\cos\theta, y = a\sin\theta \Rightarrow dy/dx = -\cot\theta$ (Su'24)
10. $\dfrac{d}{dx}(x^3 + 1) = 3x^2$ (W'24)
11. $\dfrac{d}{dx}(\sec^2 x - \tan^2 x) = 0$ (W'24)
12. $y = \log(4x-3) \Rightarrow y' = \dfrac{4}{4x-3}$ (N-W'25)

3-mark: $y = \text{cosec}^{-1}x + \sec^{-1}x$ (GTU Su'23)

Find $\frac{dy}{dx}$

$$\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} + \frac{1}{x\sqrt{x^2-1}} = \boxed{0}$$

(Because $\csc^{-1}x + \sec^{-1}x = \frac{\pi}{2}$ for $|x| > 1$, a constant.)

3-mark: $y = \log(x^2 + 2x + 4)$ (GTU Su'25)

Find $\frac{dy}{dx}$

Using chain rule: $\dfrac{d}{dx}\log(f(x)) = \dfrac{f'(x)}{f(x)}$

$f(x) = x^2 + 2x + 4, \quad f'(x) = 2x + 2$

$$\frac{dy}{dx} = \frac{2x+2}{x^2+2x+4} = \boxed{\frac{2(x+1)}{x^2+2x+4}}$$

4-mark: $y = (\tan^{-1} x)^2$ (GTU Su'23)

Prove $(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$

$$y_1 = \frac{dy}{dx} = 2\tan^{-1} x \cdot \frac{1}{1+x^2} = \frac{2\tan^{-1} x}{1+x^2}$$

Multiply both sides by $(1+x^2)$:
$(1+x^2)y_1 = 2\tan^{-1} x$

Differentiate again w.r.t. x:

$$(1+x^2)y_2 + 2x \cdot y_1 = 2 \cdot \frac{1}{1+x^2}$$

Multiply both sides by $(1+x^2)$:
$(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$ ✓

4-mark: $y = \log(\sin x)$ (GTU W'24)

Prove $y'' + \left(\frac{dy}{dx}\right)^2 + 1 = 0$

$y' = \dfrac{\cos x}{\sin x} = \cot x$

$y'' = -\csc^2 x$

LHS: $y'' + (y')^2 + 1 = -\csc^2 x + \cot^2 x + 1$

Use: $\csc^2 x - \cot^2 x = 1$, i.e. $-\csc^2 x + \cot^2 x = -1$

$= -1 + 1 = \boxed{0}$ ✓

4-mark: $y = 3e^{2x} + 4e^{-2x}$ (GTU Su'25)

Prove $y'' = 4y$

$y' = 6e^{2x} - 8e^{-2x}$

$y'' = 12e^{2x} + 16e^{-2x}$

$4y = 4(3e^{2x} + 4e^{-2x}) = 12e^{2x} + 16e^{-2x}$

$\therefore y'' = 4y$ ✓

4-mark: Kinematics $s = \frac{t^3}{3} + 3t$ (GTU Su'25)

Find v and a at $t = 3$

$v = \frac{ds}{dt} = t^2 + 3, \quad a = \frac{d^2s}{dt^2} = 2t$

At $t = 3$:

$v = 9 + 3 = \mathbf{12}$ m/s

$a = 2(3) = \mathbf{6}$ m/s^2

Unit 2 — Summary and Key Formulae

Must-Know Differentiation Cheat Sheet

First Principles: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g) \cdot g'$

Product Rule: $(uv)' = uv' + vu'$

Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

Implicit: Differentiate both sides, collect $\frac{dy}{dx}$

Parametric: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Log Diff: Take ln, differentiate, multiply by y

Velocity: $v = ds/dt; \quad a = d^2s/dt^2$

Max/Min: $f' = 0$; $f'' < 0 \Rightarrow$max; $f'' > 0 \Rightarrow$min

Unit 2 — GTU Frequency Guide

PYQ Frequency Analysis

Always in every paper:

— 3 MCQs: standard formula, tricky identity, parametric/log
— 3-mark: First principles OR implicit OR log diff
— 4-mark ($\times 2$): Second derivative proof + kinematics OR max/min

Highest frequency topics:

1. Max/Min (x^3 functions) — appears 5+ times across papers
2. Kinematics ($s = t^3$) — appears in every paper
3. Second derivative proofs ($e^x \sin x$ type) — W'22, W'23, Su'24
4. Implicit differentiation — W'22, Su'24, W'23

Unit 2 Complete!

Next: Unit 3 — Integration and its Applications

Inverse Trig & Hyperbolic Derivatives — Quick Ref

Inverse Trigonometric

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$

GTU Tip

$\sin^{-1}x + \cos^{-1}x = \pi/2 \;\Rightarrow\;$ their sum has derivative $= 0$ – classic MCQ trap!

Common Differentiation Errors — Avoid These

Most Common Student Mistakes

Wrong	**Correct**
$\frac{d}{dx}(x^n) = nx^{n-1}$ *always*	Only for constant n; use log-diff for x^x
$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}$	$= \dfrac{f'(x)}{f(x)}$ (chain rule!)
$\frac{d}{dx}a^x = xa^{x-1}$	$= a^x \ln a$
Product rule forgotten	Always apply when two functions multiply
Ignoring $+C$ in antiderivative	Always write $+C$

Unit 3: Integration and its Applications

Applied Mathematics — 4320001 / DI02000011

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Unit 3 — Table of Contents

Syllabus Coverage

Weightage: ≈14 marks (20%) | Hours: 10–11

3.1 Integration as Anti-Differentiation

What is Integration?

Definition

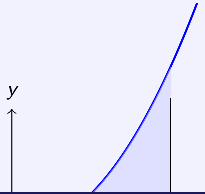
Integration is the **reverse of differentiation**.

If $\dfrac{d}{dx}[F(x)] = f(x)$, then $\displaystyle\int f(x)\,dx = F(x) + C$

where C is the **constant of integration**.

Geometric Meaning

$\displaystyle\int_a^b f(x)\,dx$ = **area** under the curve $y = f(x)$ between $x = a$ and $x = b$.



3.2 Standard Integrals

Standard Integrals Table (Part 1)

Algebraic and Exponential

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int c\,dx = cx + C \qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

GTU MCQ Examples

- ✓ $\int \frac{1}{x^2}dx = -\frac{1}{x} + C$ (W'22)
- ✓ $\int 5x^4\,dx = x^5 + C$ (Su'24)
- ✓ $\int x^2\,dx = \frac{x^3}{3} + C$ (W'24)
- ✓ $\int a^x\,dx = \frac{a^x}{\log_e a} + C$ (N-W'25)

Standard Integrals Table (Part 2)

Trigonometric

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Inverse Trig & Rational

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

3.3 Rules of Integration

Basic Integration Rules

Rules

1. **Constant:** $\displaystyle\int k\, f(x)\, dx = k \int f(x)\, dx$
2. **Sum/Difference:** $\displaystyle\int [f(x) \pm g(x)]\, dx = \int f\, dx \pm \int g\, dx$
3. **No quotient rule / product rule** for integration directly $\Rightarrow$ use substitution or integration by parts

GTU Su'25: $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

Expand: $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2 = x + 2 + \dfrac{1}{x}$

$$\int \left(x + 2 + \frac{1}{x}\right) dx = \frac{x^2}{2} + 2x + \ln|x| + C$$

3.4 Integration by Substitution

Substitution Method — Key Idea

Method

Replace an inner expression by t so the integral becomes standard.

Steps:

1. Identify $t = g(x)$ so that $\frac{dt}{dx}$ appears in the integrand.
2. Substitute $dx = \dfrac{dt}{g'(x)}$.
3. Integrate in t, then substitute back.

Key Formula

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(t)\, dt \quad [t = g(x)]$$

Substitution Example 1 (GTU Su'24)

Q: $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Step 1: Let $t = \sqrt{x}$, so $\dfrac{dt}{dx} = \dfrac{1}{2\sqrt{x}}$, i.e., $dx = 2\sqrt{x}\,dt$.

Step 2: Substitute: $\displaystyle\int \frac{\sin t}{\sqrt{x}} \cdot 2\sqrt{x}\,dt = 2\int \sin t\,dt$

Step 3: $= -2\cos t + C = \boxed{-2\cos\sqrt{x} + C}$

Substitution Example 2 (GTU W'23)

Q: $\int \frac{(1+x)e^x}{\cos^2(xe^x)}\, dx$

Step 1: Let $t = xe^x$.

$\frac{dt}{dx} = e^x + xe^x = (1+x)e^x \Rightarrow (1+x)e^x\, dx = dt$

Step 2: $\int \frac{dt}{\cos^2 t} = \int \sec^2 t\, dt$

Step 3: $= \tan t + C = \boxed{\tan(xe^x) + C}$

Substitution Example 3 (GTU W'24, Su'25)

Q: $\int \frac{\sin x \cos x}{1+\sin^2 x}\, dx$

Step 1: Let $t = \sin^2 x$, so $\frac{dt}{dx} = 2\sin x \cos x$, i.e., $\sin x \cos x\, dx = \frac{dt}{2}$

Step 2: $\int \frac{1}{1+t} \cdot \frac{dt}{2} = \frac{1}{2}\int \frac{dt}{1+t}$

Step 3: $= \frac{1}{2}\ln|1+t| + C = \boxed{\frac{1}{2}\ln(1+\sin^2 x) + C}$

Substitution: Tan/Sec type (GTU Su'24)

Q: $\displaystyle\int \frac{\tan x}{\sec x + \tan x}\,dx$

Rationalise: multiply by $(\sec x - \tan x)$:

$$\frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} = \frac{\tan x(\sec x - \tan x)}{1}$$

$$= \tan x \sec x - \tan^2 x$$

$$= \tan x \sec x - (\sec^2 x - 1) = \tan x \sec x - \sec^2 x + 1$$

$$\int (\sec x \tan x - \sec^2 x + 1)\,dx = \sec x - \tan x + x + C$$

Substitution: Definite (GTU W'24)

Q: $\int_1^e \frac{(\log x)^2}{x}\,dx$

Let $t = \log x \Rightarrow dt = \frac{1}{x}dx$

Limits: when $x = 1,\ t = 0$; when $x = e,\ t = 1$

$$\int_0^1 t^2\,dt = \left[\frac{t^3}{3}\right]_0^1 = \boxed{\frac{1}{3}}$$

GTU Su'24: $\int_0^1 \frac{x^2}{1+x^6}dx$

Let $t = x^3$, $dt = 3x^2\,dx$. Limits: $0 \to 0$, $1 \to 1$.

$$\frac{1}{3}\int_0^1 \frac{dt}{1+t^2} = \frac{1}{3}\left[\tan^{-1} t\right]_0^1 = \frac{1}{3}\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{12}}$$

3.5 Integration by Parts (IBP)

IBP — Formula and ILATE Rule

IBP Formula

$$\int u\,v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$$

ILATE — Choose u in this order

I	**L**	**A**	**T**	**E**
Inverse trig	Log	Algebraic	Trig	Exponential

Pick u = the function that appears **earlier** in ILATE.

IBP Example 1: $\int x \log x \, dx$ (GTU N-W'25)

Solution

ILATE: $u = \log x$ (L), $\quad v = x$ (A)

$$\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \log x}{2} - \frac{1}{2} \int x \, dx = \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

$$= \boxed{\frac{x^2}{4}(2 \log x - 1) + C}$$

IBP Example 2: $\int x^2 \log x \, dx$ (GTU W'23)

Solution

ILATE: $u = \log x, \; v = x^2$

$$\int x^2 \log x \, dx = \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} \, dx$$

$$= \frac{x^3 \log x}{3} - \frac{1}{3} \int x^2 \, dx = \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

$$= \boxed{\frac{x^3}{9}(3 \log x - 1) + C}$$

IBP Example 3: $\int x \sin x \, dx$ (GTU W'24)

Solution

ILATE: $u = x$ (A), $v = \sin x$ (T)

$$\int x \sin x \, dx = x(-\cos x) - \int 1 \cdot (-\cos x) \, dx$$

$$= -x \cos x + \int \cos x \, dx = \boxed{-x \cos x + \sin x + C}$$

GTU Su'25: $\int x \cos x \, dx$

$u = x, \; v = \cos x$:

$x \sin x - \int \sin x \, dx = \boxed{x \sin x + \cos x + C}$

IBP Example 4: $\int x^2 e^x \, dx$ (GTU W'22)

IBP Applied Twice

Round 1: $u = x^2$, $v = e^x$:

$$= x^2 e^x - \int 2x \, e^x \, dx$$

Round 2: For $\int 2xe^x \, dx$, $u = 2x$, $v = e^x$:

$$= 2xe^x - \int 2e^x \, dx = 2xe^x - 2e^x$$

Combine:

$$= x^2 e^x - (2xe^x - 2e^x) + C = \boxed{e^x(x^2 - 2x + 2) + C}$$

3.6 Integration by Partial Fractions

Partial Fractions — Method

When to Use

Integrand is a **rational function** $\frac{P(x)}{Q(x)}$ where degree of $P <$ degree of Q.

Standard Forms

$$\frac{A}{(x-a)} + \frac{B}{(x-b)} \quad \text{for distinct linear factors}$$

$$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} \quad \text{for repeated linear factor}$$

$$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c} \quad \text{for quadratic factor}$$

Partial Fractions Example 1 (GTU W'22)

Q: $\displaystyle\int \frac{x}{(x+1)(x+2)}\,dx$

Let $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$x = A(x+2) + B(x+1)$

Put $x = -1$: $-1 = A(1) \Rightarrow A = -1$

Put $x = -2$: $-2 = B(-1) \Rightarrow B = 2$

$$\int \left(\frac{-1}{x+1} + \frac{2}{x+2} \right) dx = \boxed{-\ln|x+1| + 2\ln|x+2| + C}$$

Partial Fractions Example 2 (GTU W'23)

Q: $\int \frac{2x+1}{(x+1)(x-3)}\,dx$

$$\frac{2x+1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$2x+1 = A(x-3) + B(x+1)$$

Put $x = -1$: $-1 = A(-4) \Rightarrow A = \frac{1}{4}$

Put $x = 3$: $7 = B(4) \Rightarrow B = \frac{7}{4}$

$$\int \left(\frac{1/4}{x+1} + \frac{7/4}{x-3}\right) dx = \boxed{\frac{1}{4}\ln|x+1| + \frac{7}{4}\ln|x-3| + C}$$

3.7 Trigonometric Integrals

Product-to-Sum Formulae

Product-to-Sum Identities

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin A \cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$

GTU W'23, W'24: $\int \sin 5x \sin 6x \, dx$

$$\sin 5x \sin 6x = \tfrac{1}{2}[\cos(5x-6x) - \cos(5x+6x)] = \tfrac{1}{2}[\cos(-x) - \cos 11x]$$
$$= \frac{1}{2}\int (\cos x - \cos 11x)\, dx = \boxed{\frac{\sin x}{2} - \frac{\sin 11x}{22} + C}$$

Split Trig Fraction (GTU W'22)

Q: $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx$

Split: $\frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} = \sec x \tan x + \csc x \cot x$

Wrong sign trap! $\int \csc x \cot x\, dx = -\csc x$

$$\int (\sec x \tan x + \csc x \cot x)\, dx = \sec x - \csc x + C$$

Warning

$\int \csc x \cot x\, dx = -\csc x + C$ (note the negative sign!)

Trig Integral: GTU Su'25

Q: $\int (\sqrt{x} + \frac{1}{\sqrt{x}})^2 dx$ (revisited with trig sense)

Expand: $x + 2 + \dfrac{1}{x}$

$$\int \left(x + 2 + \frac{1}{x} \right) dx = \frac{x^2}{2} + 2x + \ln|x| + C$$

GTU N-W'25: $\int \left(\frac{1}{x} + e^x \right) dx$

Direct standard forms:

$$\int \frac{1}{x}\, dx + \int e^x\, dx = \boxed{\ln|x| + e^x + C}$$

3.8 Definite Integrals

Definite Integral — Definition

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

where $F'(x) = f(x)$.

GTU Examples

$$\int_0^1 e^x dx = [e^x]_0^1 = e - 1 \qquad \text{(W'22, Su}$$

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8-1}{3} = \frac{7}{3} \qquad \text{(W'23)}$$

$$\int_1^3 \frac{1}{x} dx = [\ln x]_1^3 = \ln 3 - 0 = \ln 3 \qquad \text{(W'24)}$$

$$\int_2^4 (x^2 + 3x - 1)dx = \left[\frac{x^3}{3} + \frac{3x^2}{2} - x\right]_2^4 = \frac{64-8}{3} + \frac{3(16-4)}{2} - (4-2) \qquad \text{(Su'25)}$$

Definite Integral: GTU Su'25 Calculation

Q: $\int_2^4 (x^2 + 3x - 1)\, dx$

$$\left[\frac{x^3}{3} + \frac{3x^2}{2} - x\right]_2^4$$

At $x = 4$: $\frac{64}{3} + \frac{48}{2} - 4 = \frac{64}{3} + 24 - 4 = \frac{64}{3} + 20$

At $x = 2$: $\frac{8}{3} + \frac{12}{2} - 2 = \frac{8}{3} + 6 - 2 = \frac{8}{3} + 4$

Difference: $\frac{64 - 8}{3} + 20 - 4 = \frac{56}{3} + 16 = \frac{56 + 48}{3} = \boxed{\frac{104}{3}}$

GTU MCQ on Definite Integrals

MCQ Practice

Q	Answer	Paper
$\int_{-1}^{1} (3x^2 - 2x + 1)dx$	4	Su'24
$\int_{-\pi/2}^{\pi/2} \sin x \, dx$	0 (odd fn)	W'24
$\int_{0}^{1} e^x \, dx$	$e - 1$	W'22
$\int_{1}^{2} x^2 \, dx$	7/3	W'23

Odd Function Property

If $f(x)$ is odd i.e. $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x) \, dx = 0$

$$\sin(-x) = -\sin x \Rightarrow \int_{-\pi/2}^{\pi/2} \sin x \, dx = 0$$

3.9 Properties of Definite Integrals

Key Properties

Important Properties

P1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (dummy var)

P2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P4 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P5 $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Property P5 — GTU Star Question

Q: $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$ (W'22, W'23, Su'24, Su'25)

Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

Apply P5 ($a = \pi/2$, replace $x \to \pi/2 - x$):

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$$

Add: $2I = \displaystyle\int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$

$\therefore I = \boxed{\dfrac{\pi}{4}}$

Property P5 Variants (GTU W'23, Su'25)

Q: $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$

Let $I = \displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$

Apply P5 $(x \to \pi/2 - x)$: $\cot x \leftrightarrow \tan x$

$$I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\, dx$$

Add: $2I = \displaystyle\int_0^{\pi/2} 1\, dx = \frac{\pi}{2} \Rightarrow I = \boxed{\frac{\pi}{4}}$

GTU Su'24: $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$

Same method: apply P5, sum $= \pi/2$, answer $= \boxed{\pi/4}$

3.10 Area Under a Curve

Area Formula

Area Between Curve and X-axis

$$\text{Area} = \int_a^b |y|\,dx = \int_a^b |f(x)|\,dx$$

(Take absolute value if curve dips below X-axis)

Area of Circle $x^2 + y^2 = a^2$ (GTU W'24)

$y = \sqrt{a^2 - x^2}$ (upper half)

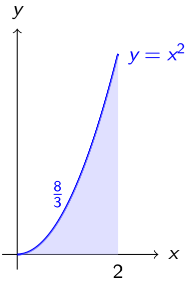
$$\text{Area} = 4\int_0^a \sqrt{a^2 - x^2}\,dx = 4 \cdot \frac{\pi a^2}{4} = \boxed{\pi a^2}$$

(Uses standard result $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$)

Area Example 1 (GTU Su'24)

Q: Area bounded by $y = x^2$, X-axis, $x = 0$ to $x = 2$

$$\text{Area} = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \boxed{\frac{8}{3}} \text{ sq. units}$$



Area Example 2 (GTU W'22)

Q: Area bounded by $y = 2x^2$, X-axis, $x = 1$ to $x = 3$

$$\text{Area} = \int_1^3 2x^2\,dx = 2\left[\frac{x^3}{3}\right]_1^3 = \frac{2}{3}(27-1) = \frac{2\times 26}{3} = \boxed{\frac{52}{3}} \text{ sq. units}$$

3.11 GTU PYQ Practice Session

MCQ Bank: 1-Mark Questions

All MCQs at a Glance

Integral	Answer	Paper
$\int x^{-2} dx$	$-1/x + C$	W'22
$\int (\log a) dx$	$x \log a + C$	W'22
$\int_0^1 e^x \, dx$	$e - 1$	W'22, Su'24
$\int \sin x \, dx$	$-\cos x + C$	W'23
$\int \frac{1}{x^2+4} dx$	$\frac{1}{2} \tan^{-1}(x/2) + C$	W'23
$\int_1^2 x^2 \, dx$	$7/3$	W'23
$\int 5x^4 \, dx$	$x^5 + C$	Su'24
$\int_{-1}^1 (3x^2 - 2x + 1) dx$	4	Su'24

MCQ Bank: More 1-Mark Questions

Continued

Integral	Answer	Paper
$\int x^2\,dx$	$x^3/3 + C$	W'24
$\int_{-\pi/2}^{\pi/2} \sin x\,dx$	0 (odd function)	W'24
$\int_1^3 \frac{1}{x}dx$	$\ln 3$	W'24
$\int \sec^2 x\,dx$	$\tan x + C$	Su'25
$\int a^x\,dx$	$a^x/\ln a + C$	N-W'25
$\int \frac{1}{x}dx$	$\ln \lvert x\rvert + C$	N-W'25

3-Mark PYQ: W'22 — Trig split

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx$$

Write as: $\displaystyle \int \sec x \tan x \, dx + \int \csc x \cot x \, dx$

$= \sec x + (-\csc x) + C = \boxed{\sec x - \csc x + C}$

3-Mark PYQ: Multiple Methods

W'23, W'24: $\int \sin 5x \sin 6x \, dx$

$= \frac{1}{2} \int [\cos(-x) - \cos 11x] dx = \frac{1}{2} \int [\cos x - \cos 11x] dx$

$= \boxed{\dfrac{\sin x}{2} - \dfrac{\sin 11x}{22} + C}$

Su'24: $\int \frac{\tan x}{\sec x + \tan x} dx$

Multiply by $(\sec x - \tan x)$: $= \int (\sec x \tan x - \sec^2 x + 1) \, dx$

$= \boxed{\sec x - \tan x + x + C}$

W'24, Su'25: $\int \frac{\sin x \cos x}{1 + \sin^2 x} dx$

Let $t = \sin^2 x$: $= \dfrac{1}{2} \int \dfrac{dt}{1+t} = \boxed{\dfrac{1}{2} \ln(1 + \sin^2 x) + C}$

4-Mark PYQ: IBP Practice Set

W'22: $\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + C$

IBP twice: $u = x^2$, then $u = 2x$
$= x^2 e^x - 2xe^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$

W'24: $\int x \sin x \, dx = -x \cos x + \sin x + C$

$u = x$, $v = \sin x$: $x(-\cos x) - \int(-\cos x) dx = -x \cos x + \sin x + C$

Su'25: $\int x \cos x \, dx = x \sin x + \cos x + C$

$u = x$, $v = \cos x$: $x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$

4-Mark PYQ: Partial Fractions Set

W'22: $\int \frac{x}{(x+1)(x+2)}dx$

PF: $\dfrac{-1}{x+1} + \dfrac{2}{x+2}$
$\Rightarrow -\ln|x+1| + 2\ln|x+2| + C$

W'23: $\int \frac{2x+1}{(x+1)(x-3)}dx$

$A = \frac{1}{4}$, $B = \frac{7}{4}$
$\Rightarrow \dfrac{1}{4}\ln|x+1| + \dfrac{7}{4}\ln|x-3| + C$

4-Mark PYQ: Property + Area Set

W'22, W'23, Su'24, Su'25: $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

Apply P5: add both forms $\rightarrow 2I = \pi/2 \Rightarrow I = \pi/4$

Su'24: Area $y = x^2$, $x = 0$ to 2

$\int_0^2 x^2\, dx = 8/3$ sq. units

W'22: Area $y = 2x^2$, $x = 1$ to 3

$2\int_1^3 x^2\, dx = \frac{2}{3}(27-1) = 52/3$ sq. units

W'24: Area of circle $x^2 + y^2 = a^2$

$4\int_0^a \sqrt{a^2 - x^2}\, dx = \pi a^2$ sq. units

Unit 3 Summary

Unit 3 — Key Takeaways

Methods Decision Tree

Direct	use standard integral table
Substitution	inner function + its derivative present
IBP	product of two different-type functions (ILATE)
Partial Fractions	rational function, degree(num) $<$ degree(den)
Product-to-Sum	$\sin mx \cdot \sin nx$, $\cos mx \cdot \cos nx$, etc.

Definite Integral Tricks

- ✓ Odd function on $[-a, a] \Rightarrow 0$
- ✓ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (use when sin/cos swap helps)
- ✓ Add I and its P5-version: $2I = \int_0^a 1\,dx = a$

Common Errors to Avoid

$\int \csc x \cot x\,dx = -\csc x + C$ (sign!)
$\int a^x\,dx = a^x / \ln a + C$ (not a^x/a)
Don't forget $+C$ in indefinite integrals

Unit 3 — GTU Weightage Summary

Marks Distribution

Topic	Marks	Frequency
MCQ (standard integrals)	1 each	2 per paper
Substitution	3	1–2 per paper
IBP	3–4	1 per paper
Partial Fractions	4	1 per paper
Definite Integral properties	3–4	1 per paper
Area under curve	4	1 per paper

Total: $\approx$ 14–16 marks out of 80

Exam-ready with all PYQ types covered!

IBP: $\int \log x \, dx$ and $\int e^x(\sin x + \cos x)\, dx$

GTU W'23, Su'24 — $\int \log x \, dx$

Let $u = \log x$, $dv = dx \Rightarrow du = \frac{1}{x} dx$, $v = x$

$\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} \, dx = x \log x - x + C$

GTU W'24 — $\int e^x(\sin x + \cos x)\, dx$

Use $\int e^x[f(x) + f'(x)]\, dx = e^x f(x) + C$

Here $f(x) = \sin x$, $f'(x) = \cos x$ ✓

$\therefore \int e^x(\sin x + \cos x)\, dx = e^x \sin x + C$

GTU W'22 — $\int \frac{\cos x}{\sin^2 x}\,dx$

Let $t = \sin x,\ dt = \cos x\,dx$
$\int \frac{1}{t^2}\,dt = -\frac{1}{t} + C = -\csc x + C$

GTU Su'25 — $\int \frac{x}{1+x^4}\,dx$

Let $t = x^2,\ dt = 2x\,dx \Rightarrow x\,dx = \frac{dt}{2}$
$\frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1}(t) + C = \frac{1}{2}\tan^{-1}(x^2) + C$

Definite: $\int_0^{\pi} x \sin x \, dx$ and $\int_0^1 \frac{x}{(1+x^2)^2} \, dx$

GTU W'23 — IBP definite

$\int_0^{\pi} x \sin x \, dx$. Let $u = x$, $dv = \sin x \, dx$
$= [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx = \pi + [\sin x]_0^{\pi} = \pi$

GTU Su'24 — Sub. in definite

$\int_0^1 \frac{x}{(1+x^2)^2} \, dx$. Let $t = 1 + x^2$, $dt = 2x \, dx$
Limits: $x = 0 \to t = 1$; $x = 1 \to t = 2$
$\frac{1}{2} \int_1^2 t^{-2} \, dt = \frac{1}{2} \left[-\frac{1}{t}\right]_1^2 = \frac{1}{2} \left(-\frac{1}{2} + 1\right) = \frac{1}{4}$

Standard Integral Identities — Must Know

Special Forms

$\int \frac{dx}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + C$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + C$
$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$
$\int \sqrt{a^2-x^2}\,dx$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
$\int e^{ax}\sin bx\,dx$	$\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$
$\int e^{ax}\cos bx\,dx$	$\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$

Reduction: $\int \frac{2x+3}{x^2+3x-4}\,dx$ (PF type, GTU W'24)

Spot the numerator = derivative of denominator

$\frac{d}{dx}(x^2+3x-4) = 2x+3$ — numerator is exactly the derivative!

$\therefore \int \frac{2x+3}{x^2+3x-4}\,dx = \ln|x^2+3x-4| + C$

Generalise: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

This is the fastest integration technique for rational functions where numerator = derivative of denominator.

GTU often tests this as a 3-mark standalone or MCQ.

Mixed PYQ Set — 3 and 4 mark (All Years)

3-Mark Practice

1. $\int \frac{\sin x}{1+\cos^2 x}\,dx$ [W'22]
2. $\int x^2 \sin x\,dx$ [Su'23]
3. $\int \frac{\log x}{x}\,dx$ [W'23]
4. $\int_0^{\pi/2} \sin^3 x \cos^2 x\,dx$ [N-W'25]

4-Mark Practice

1. $\int \frac{x^2+1}{(x+2)(x-3)}\,dx$ [partial fractions]
2. $\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$ [W'24 area-type]
3. Area bounded by $y = x^3$, x-axis, $x = 0$ to $x = 2$ [Su'25]
4. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \pi^2/4$ [property P5]

Unit 3 Complete!

Next: Unit 4 — Differential Equations

Unit 4: Differential Equations

Applied Mathematics — 4320001 / DI02000011

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April 2, 2026

Unit 4 — Table of Contents

Syllabus Coverage

Weightage: ≈12 marks (17%) | Hours: 9–10

4.1 Introduction to Differential Equations

What is a Differential Equation?

Definition

A **differential equation (DE)** is an equation involving an unknown function and its derivatives.

Examples

$$\frac{dy}{dx} = x^2 \qquad \text{(1st order)}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \qquad \text{(2nd order)}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + 1 = 0 \qquad \text{(2nd order, degree 3)}$$

General Solution vs Particular Solution

General: Contains arbitrary constant C.
Particular: C fixed using an initial/boundary condition.

4.2 Order and Degree

Order and Degree — Definitions

Order

The **order** of a DE is the order of the **highest derivative** present.

Degree

The **degree** is the power of the **highest-order derivative** after clearing fractions/radicals.

Trick

First find the **highest derivative**, then check its **power**.
Degree is undefined if the DE has sin, cos, log *of derivatives.*

Order & Degree — GTU MCQ Practice

MCQ Bank

Equation	Order	Degree
$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$	2	1 *(W'22)*
$\left(\frac{d^3x}{dy^3}\right) + \left(\frac{dy}{dx}\right)^4 + 5y = 0$	3	1 *(W'23)*
$\left(\frac{dy}{dx}\right)^2 + 4y = x$	1	2 *(Su'24)*
$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + 1 = 0$	2	3 *(W'24)*
$\left(\frac{d^3y}{dx^3}\right) + \left(\frac{d^2y}{dx^2}\right) + x\cos y = 0$	3	1 *(Su'25)*
$\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$	2	1 *(N-W'25)*

Linear DE: $\frac{dy}{dx} + P(x)\,y = Q(x)$

$\text{IF} = e^{\int P(x)\,dx}$

All IF MCQs

DE	P(x)	IF	Paper
$\frac{dy}{dx} + y = 3x$	1	e^x	W'22
$\frac{dy}{dx} + \frac{y}{x} = 1$	$\frac{1}{x}$	$e^{\ln x} = x$	W'23
$\frac{dy}{dx} + 3y = x$	3	e^{3x}	Su'24
$\frac{dy}{dx} + y = 1$	1	e^x	W'24
$\frac{dy}{dx} + \frac{2}{x}y = e^x$	$\frac{2}{x}$	$e^{2\ln x} = x^2$	N-W'25

4.3 Variable Separable Method

Variable Separable — Method

When to Use

The DE can be written as $f(y)\,dy = g(x)\,dx$ — all y on left, all x on right.

Steps

1. Rearrange to separate variables: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$
2. Write as $h(y)\,dy = g(x)\,dx$
3. Integrate both sides: $\displaystyle\int h(y)\,dy = \int g(x)\,dx + C$
4. Simplify to get the solution.

Var. Sep. Example 1 (GTU W'22)

Q: Solve $\dfrac{dy}{dx} + x^2 e^{-y} = 0$

Rearrange: $\dfrac{dy}{dx} = -x^2 e^{-y}$

Separate: $e^y\,dy = -x^2\,dx$

Integrate: $\displaystyle\int e^y\,dy = -\int x^2\,dx$

$e^y = -\dfrac{x^3}{3} + C$

General Solution: $\boxed{e^y = C - \dfrac{x^3}{3}}$

Var. Sep. Example 2 (GTU Su'24)

Q: Solve $\dfrac{dy}{dx} - 3x^2 e^{-y} = 0$

Rearrange: $e^y\,dy = 3x^2\,dx$

Integrate: $e^y = x^3 + C$

General Solution: $\boxed{e^y = x^3 + C}$

Compare W'22 vs Su'24

W'22: $+x^2 e^{-y} = 0$ sign $\rightarrow$ $e^y = C - x^3/3$
Su'24: $-3x^2 e^{-y} = 0$ sign $\rightarrow$ $e^y = x^3 + C$
Key: Watch the sign carefully before separating.

Var. Sep. Example 3 (GTU W'23)

Q: Solve $xy\,dy = (x+1)(y+1)\,dx$

Separate: $\dfrac{y}{y+1}\,dy = \dfrac{(x+1)}{x}\,dx$

Write LHS: $\dfrac{y}{y+1} = 1 - \dfrac{1}{y+1}$

Write RHS: $\dfrac{x+1}{x} = 1 + \dfrac{1}{x}$

Integrate: $y - \ln|y+1| = x + \ln|x| + C$

General Solution: $\boxed{y - x = \ln|x| + \ln|y+1| + C}$

Q: Solve $x\,dy + y\,dx = xy\,dy$

Recognise: $x\,dy + y\,dx = d(xy)$

So: $d(xy) = xy\,dy$

Let $v = xy$: $dv = v\,dy \Rightarrow \dfrac{dv}{v} = dy$

Integrate: $\ln|v| = y + C$

General Solution: $\boxed{\ln|xy| = y + C}$ i.e., $xy = Ae^y$

Var. Sep. Example 5 (GTU Su'25)

Q: Solve $\dfrac{dy}{dx} = 1 + x + y + xy$

Factor RHS: $1 + x + y + xy = (1 + x)(1 + y)$

Separate: $\dfrac{dy}{1 + y} = (1 + x)\,dx$

Integrate: $\ln|1 + y| = x + \dfrac{x^2}{2} + C$

General Solution: $\boxed{1 + y = Ae^{x + x^2/2}}$

Tip

Always look for **factoring** before separating.
$1 + x + y + xy = (1 + x)(1 + y)$ is a classic exam trick.

Var. Sep. Example 6 (GTU W'24)

Q: Solve $xy\,dx + (1+x^2)\,dy = 0$

Divide by $y(1+x^2)$: $\dfrac{x}{1+x^2}\,dx + \dfrac{dy}{y} = 0$

Integrate: $\dfrac{1}{2}\ln(1+x^2) + \ln|y| = \ln C$

$\ln\left[y\sqrt{1+x^2}\right] = \ln C$

General Solution: $\boxed{y\sqrt{1+x^2} = C}$

Var. Sep. Example 7 (GTU Su'24)

Q: Solve $\dfrac{dy}{dx} + \tan x \tan y = 0$

Separate: $\dfrac{dy}{\tan y} = -\tan x\, dx$

i.e., $\cot y\, dy = -\tan x\, dx$

Integrate: $\ln|\sin y| = \ln|\cos x| + \ln C$

$\ln|\sin y| = \ln|C\cos x|$

General Solution: $\boxed{\sin y = C\cos x}$

Var. Sep. Example 8 (GTU W'23)

Q: Solve $y\dfrac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$

Factor: $\sqrt{1 + x^2 + y^2 + x^2y^2} = \sqrt{(1 + x^2)(1 + y^2)}$

Separate: $\dfrac{y\,dy}{\sqrt{1 + y^2}} = \sqrt{1 + x^2}\,dx$

LHS: let $u = 1 + y^2$, $du = 2y\,dy \Rightarrow \dfrac{du}{2\sqrt{u}} \Rightarrow \sqrt{1 + y^2}$

RHS: $\displaystyle\int \sqrt{1 + x^2}\,dx = \frac{x\sqrt{1 + x^2}}{2} + \frac{\sinh^{-1} x}{2} + C$

Solution: $\boxed{\sqrt{1 + y^2} = \dfrac{x\sqrt{1 + x^2}}{2} + \dfrac{\ln(x + \sqrt{1 + x^2})}{2} + C}$

4.4 Substitution Method

When to Substitute

Reducible DEs

Form	**Substitution**
$\dfrac{dy}{dx} = f(ax + by + c)$	$v = ax + by + c$
Homogeneous: $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$	$y = vx$
Bernoulli: $\dfrac{dy}{dx} + Py = Qy^n$	$v = y^{1-n}$
Mixed product	$v = xy$ or $v = x + y$

Substitution Example 1 (GTU W'22, W'23)

Q: Solve $(x+y+1)^2\dfrac{dy}{dx}=1$

Let $v=x+y+1 \Rightarrow \dfrac{dv}{dx}=1+\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

Substitute: $v^2\left(\dfrac{dv}{dx}-1\right)=1 \Rightarrow v^2\dfrac{dv}{dx}=v^2+1$

Separate: $\dfrac{v^2}{v^2+1}\,dv=dx \Rightarrow \left(1-\dfrac{1}{v^2+1}\right)dv=dx$

Integrate: $v-\tan^{-1}v=x+C$

Back-sub: $(x+y+1)-\tan^{-1}(x+y+1)=x+C$

Solution: $\boxed{y+1-\tan^{-1}(x+y+1)=C}$

Substitution Example 2 — Homogeneous (GTU Su'23)

Q: Solve $xy\,dx - (y^2 + x^2)\,dy = 0$

Rewrite: $\dfrac{dx}{dy} = \dfrac{y^2 + x^2}{xy} = \dfrac{y}{x} + \dfrac{x}{y}$

Let $x = vy \Rightarrow \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Substitute: $v + y\dfrac{dv}{dy} = \dfrac{1}{v} + v \Rightarrow y\dfrac{dv}{dy} = \dfrac{1}{v}$

Separate: $v\,dv = \dfrac{dy}{y}$

Integrate: $\dfrac{v^2}{2} = \ln|y| + C \Rightarrow \dfrac{x^2}{2y^2} = \ln|y| + C$

Solution: $\boxed{x^2 = 2y^2(\ln|y| + C)}$

Substitution Example 3 (GTU Su'23)

Q: Solve $(x + y)\, dy = dx$

Rewrite as: $\dfrac{dx}{dy} = x + y \Rightarrow \dfrac{dx}{dy} - x = y$

This is **linear in** x with $P = -1$, $Q = y$.

IF $= e^{\int(-1)dy} = e^{-y}$

Solution: $x\,e^{-y} = \displaystyle\int y\,e^{-y}\,dy = -ye^{-y} - e^{-y} + C$ (IBP)

$x\,e^{-y} = -(y+1)e^{-y} + C$

Solution: $\boxed{x = -(y+1) + Ce^{y}}$ i.e., $x + y + 1 = Ce^{y}$

4.5 Linear Differential Equations

Linear DE — Method

Standard Form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Steps to Solve

1. Find IF $= e^{\int P(x)\,dx}$
2. Multiply both sides by IF
3. LHS becomes $\dfrac{d}{dx}[y \cdot \text{IF}]$
4. Integrate: $y \cdot \text{IF} = \displaystyle\int Q \cdot \text{IF}\,dx + C$

Linear DE Example 1 (GTU W'22)

Q: Solve $\dfrac{dy}{dx} + \dfrac{4x}{1+x^2}\, y = \dfrac{1}{(1+x^2)^2}$

$$P = \frac{4x}{1+x^2}, \quad Q = \frac{1}{(1+x^2)^2}$$

$$\int P\,dx = \int \frac{4x}{1+x^2}\,dx = 2\ln(1+x^2)$$

$$\text{IF} = e^{2\ln(1+x^2)} = (1+x^2)^2$$

$$y(1+x^2)^2 = \int \frac{(1+x^2)^2}{(1+x^2)^2}\,dx = \int 1\,dx = x + C$$

Solution: $\boxed{y = \dfrac{x+C}{(1+x^2)^2}}$

Q: Solve $\frac{dy}{dx} + y\tan x = \sec x$

$P = \tan x, \quad Q = \sec x$

$\int P\,dx = \int \tan x\,dx = \ln|\sec x|$

$\text{IF} = e^{\ln|\sec x|} = \sec x$

$y\sec x = \int \sec x \cdot \sec x\,dx = \int \sec^2 x\,dx = \tan x + C$

Solution: $\boxed{y\sec x = \tan x + C}$ i.e., $y = \sin x + C\cos x$

Q: Solve $x\dfrac{dy}{dx} - y = x^2$

Divide by x: $\dfrac{dy}{dx} - \dfrac{y}{x} = x$

$P = -\dfrac{1}{x}, \quad Q = x$

$\text{IF} = e^{-\ln x} = \dfrac{1}{x}$

$\dfrac{y}{x} = \displaystyle\int x \cdot \frac{1}{x}\, dx = \int 1\, dx = x + C$

Solution: $\boxed{y = x^2 + Cx}$

Linear DE Example 4 (GTU Su'24)

Q: Solve $\dfrac{dy}{dx} + 2y = 3e^x$

$P = 2$, IF $= e^{2x}$

$$ye^{2x} = \int 3e^x \cdot e^{2x}\,dx = 3\int e^{3x}\,dx = e^{3x} + C$$

Solution: $\boxed{y = e^x + Ce^{-2x}}$

GTU W'24: $\frac{dy}{dx} + y = 1$ (for IF MCQ)

$P = 1$, IF $= e^x$. Solution: $ye^x = \int e^x\,dx = e^x + C \Rightarrow y = 1 + Ce^{-x}$

Linear DE Example 5 (GTU Su'23)

Q: Solve $(1+x^2)\dfrac{dy}{dx} + 2xy = \cos x$

Divide: $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}y = \dfrac{\cos x}{1+x^2}$

$\text{IF} = e^{\int \frac{2x}{1+x^2}\,dx} = e^{\ln(1+x^2)} = 1+x^2$

$y(1+x^2) = \displaystyle\int \frac{\cos x}{1+x^2}(1+x^2)\,dx = \int \cos x\,dx = \sin x + C$

Solution: $\boxed{y(1+x^2) = \sin x + C}$

Linear DE Example 6 (GTU W'24, N-W'25 IF MCQ)

Q: Solve $(x^2+1)\dfrac{dy}{dx} + 2xy = e^x$ (W'24 3-mark)

Divide: $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}y = \dfrac{e^x}{1+x^2}$

$\text{IF} = (1+x^2)$

$y(1+x^2) = \displaystyle\int e^x\,dx = e^x + C$

Solution: $\boxed{y = \dfrac{e^x + C}{1+x^2}}$

GTU Su'23: $\frac{dy}{dx} + \frac{2y}{x} = \sin x$

IF$= x^2$. $yx^2 = \int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

$y = \dfrac{-x^2\cos x + 2x\sin x + 2\cos x + C}{x^2}$

4.6 Initial Value Problems (IVP)

IVP — What and How

Initial Value Problem

A DE together with a condition $y(x_0) = y_0$ is called an **IVP**.

The condition fixes the constant C in the general solution.

Steps

1. Solve the DE to get **general solution** (with C).
2. Substitute the initial condition $y(x_0) = y_0$.
3. Solve for C and write the **particular solution**.

IVP Example 1 (GTU W'22)

Q: Solve $\frac{dy}{dx} + y = e^x$, $y(0) = 1$

Step 1 — Solve: $P = 1$, IF $= e^x$

$$ye^x = \int e^x \cdot e^x\,dx = \int e^{2x}\,dx = \frac{e^{2x}}{2} + C$$

General: $y = \frac{e^x}{2} + Ce^{-x}$

Step 2 — Apply $y(0) = 1$:

$$1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Particular Solution: $\boxed{y = \frac{e^x}{2} + \frac{e^{-x}}{2} = \cosh x}$

IVP Example 2 (GTU Su'24)

Q: Solve $dy + 4xy^2\,dx = 0$, $y(0) = 1$

Separate: $\dfrac{dy}{y^2} = -4x\,dx$

Integrate: $-\dfrac{1}{y} = -2x^2 + C \Rightarrow \dfrac{1}{y} = 2x^2 - C$

Apply $y(0) = 1$**:** $\dfrac{1}{1} = 0 - C \Rightarrow C = -1$

$$\frac{1}{y} = 2x^2 + 1$$

Particular Solution: $\boxed{y = \dfrac{1}{2x^2 + 1}}$

IVP Example 3 (GTU W'24)

Q: Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = 0,\ y(2) = 1$

Separate: $\dfrac{dy}{y} = -\dfrac{dx}{x}$

Integrate: $\ln|y| = -\ln|x| + \ln C \Rightarrow xy = C$

Apply $y(2) = 1$: $2 \times 1 = C \Rightarrow C = 2$

Particular Solution: $\boxed{xy = 2}$ i.e., $y = \dfrac{2}{x}$

4.7 GTU PYQ Practice Session

MCQ Bank — Order, Degree & IF

All 1-Mark MCQs at a Glance

Question	Answer	Paper
Order & degree of $y'' - 5y' + 6y = 0$	2, 1	W'22
IF of $y' + y = 3x$	e^x	W'22
Order of $(x''') + (y')^4 + 5y = 0$	3	W'23
IF of $y' + y/x = 1$	x	W'23
Order of $(y')^2 + 4y = x$	1	Su'24
IF of $y' + 3y = x$	e^{3x}	Su'24
Order & degree of $(y'')^3 + y' + 1 = 0$	2, 3	W'24
IF of $y' + y = 1$	e^x	W'24
Order & degree of $y''' + y'' + x\cos y = 0$	3, 1	Su'25
Order of $y'' + y' + xy = 0$	2	N-W'25
IF of $y' + (2/x)y = e^x$	x^2	N-W'25

3-Mark PYQ Set (Part 1)

W'22: $y' + x^2 e^{-y} = 0$

$e^y\,dy = -x^2\,dx \Rightarrow e^y = C - x^3/3$

W'23: $xy\,dy = (x+1)(y+1)\,dx$

$(1 - \frac{1}{y+1})dy = (1 + \frac{1}{x})dx \Rightarrow y - \ln|y+1| = x + \ln|x| + C$

W'24: $(x^2+1)y' + 2xy = e^x$

IF$= (1+x^2)$. $y(1+x^2) = e^x + C$

3-Mark PYQ Set (Part 2)

Su'24: $y' - 3x^2 e^{-y} = 0$

$e^y = x^3 + C$

Su'25: $x\,dy + y\,dx = xy\,dy$

$d(xy) = xy\,dy \Rightarrow \ln|xy| = y + C$

Su'23: $(x+y)\,dy = dx$

Linear in x: IF$= e^{-y}$. $\Rightarrow x + y + 1 = Ce^y$

4-Mark PYQ: Linear DE (Part 1)

W'22: $y' + \frac{4x}{1+x^2} y = \frac{1}{(1+x^2)^2}$

IF$= (1+x^2)^2$. $y(1+x^2)^2 = x + C \Rightarrow y = \frac{x+C}{(1+x^2)^2}$

W'24: $y' + y\tan x = \sec x$

IF$= \sec x$. $y\sec x = \tan x + C \Rightarrow y = \sin x + C\cos x$

Su'24: $y' + 2y = 3e^x$

IF$= e^{2x}$. $ye^{2x} = e^{3x} + C \Rightarrow y = e^x + Ce^{-2x}$

4-Mark PYQ: Linear DE (Part 2)

Su'23: $(1+x^2)y' + 2xy = \cos x$

IF$= 1+x^2$. $y(1+x^2) = \sin x + C$

Su'25: $xy' - y = x^2$

$y' - y/x = x$. IF$= 1/x$. $y/x = x + C \Rightarrow y = x^2 + Cx$

Su'23: $y' + 2y/x = \sin x$

IF$= x^2$. $yx^2 = \int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$

4-Mark PYQ: Substitution Set

W'22, W'23: $(x+y+1)^2 y' = 1$

Let $v = x+y+1$. $v - \tan^{-1} v = x + C \Rightarrow y + 1 - \tan^{-1}(x+y+1) = C$

Su'23: $(x+y)\, dy = dx$

Treat as linear in x: $dx/dy - x = y$. IF$= e^{-y}$. Solution: $x + y + 1 = Ce^{y}$

Su'23: $xy\, dx - (y^2 + x^2)\, dy = 0$

Let $x = vy$. $v\, dv = dy/y \Rightarrow x^2 = 2y^2(\ln|y| + C)$

4-Mark PYQ: IVP Set

W'22: $y' + y = e^x,\ y(0) = 1$

$y = e^x/2 + Ce^{-x}$. $y(0) = 1 \Rightarrow C = 1/2 \Rightarrow y = \cosh x$

Su'24: $dy + 4xy^2\,dx = 0,\ y(0) = 1$

$1/y = 2x^2 - C$. $y(0) = 1 \Rightarrow C = -1 \Rightarrow y = 1/(2x^2 + 1)$

W'24: $y' + y/x = 0,\ y(2) = 1$

$xy = C$. $y(2) = 1 \Rightarrow C = 2 \Rightarrow xy = 2$

4-Mark PYQ: More Separable

Su'24: $y' + \tan x \cdot \tan y = 0$

$\cot y \, dy = -\tan x \, dx \Rightarrow \ln|\sin y| = \ln|\cos x| + C \Rightarrow \sin y = A\cos x$

Su'25: $y' = 1 + x + y + xy$

$(1+x)(1+y)$: $\ln|1+y| = x + x^2/2 + C \Rightarrow 1 + y = Ae^{x+x^2/2}$

W'24: $xy\,dx + (1+x^2)\,dy = 0$

$\frac{x}{1+x^2}dx + \frac{dy}{y} = 0 \Rightarrow y\sqrt{1+x^2} = C$

W'23: $yy' = \sqrt{(1+x^2)(1+y^2)}$

Separate $y/dy/\sqrt{1+y^2} = \sqrt{1+x^2}dx$. Integrate both sides.

Unit 4 Summary

Unit 4 — Methods Decision Tree

Which Method to Use?

Can separate $f(y)\,dy = g(x)\,dx$	Variable Separable
Form $v = ax + by + c$ appears	Substitution
Homogeneous $dy/dx = f(y/x)$	Let $y = vx$
Linear: $y' + Py = Q$	IF method
Condition $y(x_0) = y_0$ **given**	Solve $\rightarrow$ apply IVP

IF Formulae to Memorise

$P(x)$	$\int P\,dx$	IF
k (const.)	kx	e^{kx}
$1/x$	$\ln x$	x
$2x/(1+x^2)$	$\ln(1+x^2)$	$1+x^2$
$\tan x$	$\ln\lvert\sec x\rvert$	$\sec x$
$2/x$	$2\ln x$	x^2

Unit 4 — GTU Weightage

Marks Distribution

Topic	Marks	Frequency
Order & Degree (MCQ)	1+1	2 per paper
IF of linear DE (MCQ)	1+1	2 per paper
Variable Separable	3	1 per paper
Linear DE / Substitution	4	2 per paper
IVP	4	1 per paper (may combine)

Common Errors

✗ Forgetting $+C$ after integrating
✗ Wrong sign when separating (check $\pm$ carefully)
✗ Applying IVP to general solution — substitute, don't differentiate again
✗ IF: $\int 1/x\,dx = \ln x$, not $1/x^2$

Additional PYQ Practice

High-frequency exam questions

Variable Separable — Extended Set 1

Practice

1. $\dfrac{dy}{dx} = e^{2x-3y}$ [W'22: separate $e^{3y}dy = e^{2x}dx$; $\frac{e^{3y}}{3} = \frac{e^{2x}}{2} + C$]
2. $\dfrac{dy}{dx} = \dfrac{y}{x}$ [Su'23: $\ln y = \ln x + C_1 \Rightarrow y = kx$]
3. $x\,dy + y\,dx = 0$ [W'24: $xy = C$]
4. $(1 + x^2)dy = (1 + y^2)dx$ [Su'25: $\tan^{-1} y - \tan^{-1} x = C$]
5. $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos y}$ $\cos y\,dy = \sin x\,dx \Rightarrow \sin y = -\cos x + C$

Variable Separable — Extended Set 2 (W'23, Su'25)

W'23: $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0 \Rightarrow \ln|\tan x| + \ln|\tan y| = C \Rightarrow \tan x \tan y = k$$

Su'25: $\dfrac{dy}{dx} = \dfrac{2-y}{x}$

$$\frac{dy}{2-y} = \frac{dx}{x} \Rightarrow -\ln|2-y| = \ln x + C_1 \Rightarrow x(2-y) = k$$

W'22: $(1-x^2)\dfrac{dy}{dx} + xy = ax$

This is linear! Rewrite as $y' + \frac{x}{1-x^2} y = \frac{ax}{1-x^2}$ and use IF method.

Homogeneous DE — Extended Examples

GTU Su'23: $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$

Let $y = vx$: $v + x\frac{dv}{dx} = \frac{1+v^2}{v} \Rightarrow x\frac{dv}{dx} = \frac{1}{v}$
$v\,dv = \frac{dx}{x} \Rightarrow \frac{v^2}{2} = \ln x + C \Rightarrow y^2 = 2x^2(\ln x + C)$

GTU W'24: $(x^2 - y^2)dx + 2xy\,dy = 0$

Homogeneous; let $y = vx$:
$(1 - v^2) + 2v(v + x\frac{dv}{dx}) = 0 \Rightarrow (1 + v^2) + 2vx\frac{dv}{dx} = 0$
Separate: $x^2 + y^2 = Cx$

Linear DE — Extended Set

GTU N-W'25: $\dfrac{dy}{dx} + y\cot x = \sin 2x$

$\text{IF} = e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$
$y \sin x = \int \sin x \cdot \sin 2x\,dx = 2\int \sin^2 x \cos x\,dx = \frac{2\sin^3 x}{3} + C$
$\therefore y = \frac{2\sin^2 x}{3} + \frac{C}{\sin x}$

GTU Su'23: $x\dfrac{dy}{dx} + y = x^3$

Divide by x: $\frac{dy}{dx} + \frac{y}{x} = x^2$; $\text{IF} = x$
$y \cdot x = \int x \cdot x^2\,dx = \frac{x^4}{4} + C \Rightarrow y = \frac{x^3}{4} + \frac{C}{x}$

IVP Extended — Applied Problems

GTU W'23: Growth/Decay Model

Newton's law of cooling / growth: $\frac{dy}{dt} = ky$
Solution: $y = y_0 e^{kt}$
If population doubles in 5 yr ($y_0 \to 2y_0$): $2 = e^{5k} \Rightarrow k = \frac{\ln 2}{5}$
Population at $t = 10$: $y = y_0 \cdot 4$

GTU Su'24: $\frac{dy}{dx} = 2xy$; $y(0) = 3$

Separate: $\frac{dy}{y} = 2x\,dx \Rightarrow \ln|y| = x^2 + C$
$y = Ae^{x^2}$; apply $y(0) = 3$: $A = 3$; $\therefore y = 3e^{x^2}$

Substitution: More GTU Examples

GTU W'24: $\dfrac{dy}{dx} = (4x + y + 1)^2$

Let $v = 4x + y + 1$, $\frac{dv}{dx} = 4 + \frac{dy}{dx}$

$\frac{dv}{dx} = 4 + v^2 \Rightarrow \frac{dv}{4+v^2} = dx$

$\frac{1}{2}\tan^{-1}\frac{v}{2} = x + C \Rightarrow \tan^{-1}\frac{4x+y+1}{2} = 2x + C$

GTU N-W'25: $\dfrac{dy}{dx} = \cos(x + y)$

Let $v = x + y$: $\frac{dv}{dx} = 1 + \cos v \Rightarrow \int \frac{dv}{1+\cos v} = \int dx$

Use $1 + \cos v = 2\cos^2\frac{v}{2}$: $\int \frac{1}{2}\sec^2\frac{v}{2}\,dv = dx$

$\tan\frac{x+y}{2} = x + C$

Bernoulli's DE — Special Type

Form: $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

Divide by y^n: let $v = y^{1-n}$, $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$
Reduces to linear: $\frac{dv}{dx} + (1-n)Pv = (1-n)Q$

Example: $\frac{dy}{dx} + \frac{y}{x} = xy^2$

$n = 2$; divide by y^2; let $v = y^{-1}$
$\frac{dv}{dx} - \frac{v}{x} = -x$
$\text{IF} = e^{-\ln x} = \frac{1}{x}$; $\frac{v}{x} = -\ln x + C \Rightarrow \frac{1}{xy} = -\ln x + C$

Exact Differential Equations (Introductory)

Exact DE: $M\,dx + N\,dy = 0$

Condition: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution: $F(x, y) = \int M\,dx + g(y) = C$ where $\frac{\partial F}{\partial y} = N$

Example: $(2xy + 3)\,dx + (x^2 + 4y)\,dy = 0$

Check: $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ ✓ Exact!
$F = \int (2xy + 3)\,dx = x^2 y + 3x + g(y)$
$\frac{\partial F}{\partial y} = x^2 + g' = N = x^2 + 4y \Rightarrow g' = 4y \Rightarrow g = 2y^2$
Solution: $x^2 y + 3x + 2y^2 = C$

Revision Flash Points — Unit 4

Key Formulas to Memorise

VS	Separate variables, integrate both sides, apply IVP
Homogeneous	$y = vx$; reduce to VS in v
Linear	$\text{IF} = e^{\int P\,dx}$; solution $y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$
IVP	Solve general, substitute (x_0, y_0) to find C
Growth/Decay	$y = y_0 e^{kt}$; if doubles: $k = \frac{\ln 2}{T}$
Bernoulli	Divide by y^n, let $v = y^{1-n}$, linear in v
Exact	Check $M_y = N_x$; integrate M w.r.t. x

GTU Frequency Analysis — All Paper Types

Highest-Frequency Questions

Question Type	Marks	Papers
Order & Degree MCQ	1+1	Every paper
IF of $dy/dx + y\cot x = \ldots$ (MCQ)	1	W'23, Su'24, W'24
VS: Trigonometric types	3–4	Every paper
Linear DE with e^x or $\ln x$	4	Every paper
IVP (any method)	4	W'22, Su'24, W'24
Homogeneous DE	4	Su'23, W'24, N-W'25

Exam Strategy

- Always show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for exact DE check
- Write IF formula before applying
- Verify IVP by substituting back

Unit 4 — End Slide

Unit 4 Complete!

Next: Unit 5 — Statistics

Unit 5: Statistics

Applied Mathematics — 4320001 / DI02000011

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April 2, 2026

Unit 5 — Table of Contents

Syllabus Coverage

Weightage: ≈12 marks (17%) | Hours: 9–10

5.1 Introduction & Key Definitions

Why Statistics?

Key Measures in This Unit

Mean ($\bar{x}$)	Central tendency (average)
Mean Deviation (MD)	Average distance from mean
Standard Deviation (σ)	Spread / dispersion
Variance (σ^2)	Square of SD
Coeff. of Variation (CV)	Relative spread (%)
Range	Max − Min

Notation

n = number of observations f_i = frequency x_i = value / midpoint
$N = \sum f_i$ (total frequency) $\bar{x}$ = mean

All MCQs at a Glance

Question	Answer	Paper
Mean of first 5 natural numbers	3	W'22, W'23
Mean of $11, x, 19, 21, y, 29 = 20$; $x + y =?$	40	W'22
Mean of $39, 23, 58, 47, 50, 16, 61$	47	W'23
Mean of first 10 natural numbers	5.5	Su'24
Range of $17, 15, 25, 34, 32$	19	Su'24
Mean of $1, 3, 5, 7, 9$	5	W'24
Mean of $15, 7, 6, a, 3 = 4$; $a =?$	-11	W'24
Mean of first 5 even natural numbers	6	Su'25

5.2 Mean: Direct Method

Mean Formulas

Ungrouped Data

$$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Grouped (Frequency Distribution)

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{\sum f_i x_i}{N}$$

where $x_i =$ **midpoint** of class $= \dfrac{\text{lower} + \text{upper}}{2}$

Combined Mean

$$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Mean Example 1 — Ungrouped (GTU MCQ practice)

Q: Mean of $1, 3, 5, 7, 9$

$$\bar{x} = \frac{1+3+5+7+9}{5} = \frac{25}{5} = \boxed{5}$$ (W'24)

Q: Mean of first 5 natural numbers

$$1, 2, 3, 4, 5 \Rightarrow \bar{x} = \frac{15}{5} = \boxed{3}$$ (W'22, W'23)

Q: If mean of $15, 7, 6, a, 3$ is 4, find a

$$\frac{15+7+6+a+3}{5} = 4 \Rightarrow 31 + a = 20 \Rightarrow \boxed{a = -11}$$ (W'24)

Mean Example 2 — Frequency Table (GTU W'22)

Q: Find mean x_i: 92,93,97,98,102,104 f_i: 3,2,3,2,6,4

x_i	f_i	$f_i x_i$
92	3	276
93	2	186
97	3	291
98	2	196
102	6	612
104	4	416
Total	**20**	**1977**

$$\bar{x} = \frac{1977}{20} = \boxed{98.85}$$

Mean Example 3 — Grouped (GTU Su'24)

Q: Find mean (Age 20–59, staff data)

Class	f_i	mid x_i	$f_i x_i$
20–24	5	22	110
25–29	7	27	189
30–34	9	32	288
35–39	11	37	407
40–44	10	42	420
45–49	8	47	376
50–54	6	52	312
55–59	4	57	228
Total	**60**		**2330**

$$\bar{x} = \frac{2330}{60} = \boxed{38.83}$$

Combined Mean (GTU Su'24)

Q: 25 obs. with mean 50 and 75 obs. with mean 60 — find combined mean

$$\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{25 \times 50 + 75 \times 60}{25 + 75}$$

$$= \frac{1250 + 4500}{100} = \frac{5750}{100} = \boxed{57.5}$$

Q: If mean of $15, 7, 6, a, 3$ is 7, find a (W'22)

$$\frac{31 + a}{5} = 7 \Rightarrow 31 + a = 35 \Rightarrow \boxed{a = 4}$$

5.3 Mean: Assumed Mean & Step Deviation

Shortcut methods for grouped frequency data

Assumed Mean & Step Deviation — Formulas

Assumed Mean Method

Choose assumed mean A (usually central value).

$d_i = x_i - A$ then $\bar{x} = A + \dfrac{\sum f_i d_i}{N}$

Step Deviation Method

Choose class width h and let $u_i = \dfrac{x_i - A}{h}$.

$\bar{x} = A + h \cdot \dfrac{\sum f_i u_i}{N}$

When to use

Use step deviation when all class widths are equal — it gives smaller numbers to work with.

Step Deviation Example — GTU W'22 (4-mark)

Find mean marks: $A = 38$, $h = 5$

Class	f	x	$u = \frac{x-38}{5}$	fu
21 – 25	8	23	-3	-24
26 – 30	10	28	-2	-20
31 – 35	24	33	-1	-24
36 – 40	30	38	0	0
41 – 45	12	43	1	12
46 – 50	16	48	2	32
Total	**100**			**-24**

$$\bar{x} = 38 + 5 \times \frac{-24}{100} = 38 - 1.2 = \boxed{36.8}$$

Assumed Mean Example — GTU W'23

Find mean score; class 0–120, $A = 60$, $h = 20$

Class	f	x	$u = \frac{x-60}{20}$	fu
0 – 20	26	10	-2.5	-65
20 – 40	31	30	-1.5	-46.5
40 – 60	35	50	-0.5	-17.5
60 – 80	42	70	0.5	21
80 – 100	82	90	1.5	123
100–120	71	110	2.5	177.5
Total	**287**			**192.5**

$$\bar{x} = 60 + 20 \times \frac{192.5}{287} =$$

$60 + 13.43 = \boxed{73.43}$

Mean$= 19$; f_i: 2,4,7,f,8,4,3 for x_i: 6,10,14,18,24,28,30 (3 marks)

$N = 28 + f \quad \sum f_i x_i = 12 + 40 + 98 + 18f + 192 + 112 + 90 = 544 + 18f$

$$\bar{x} = 19 \Rightarrow \frac{544 + 18f}{28 + f} = 19$$

$544 + 18f = 532 + 19f \Rightarrow \boxed{f = 12}$

Missing frequency (W'24): mean$= 100$

Use same method: set up $\bar{x} = \dfrac{\sum fx}{N} = 100$ and solve for unknown x.

Mean from Freq Table — GTU Su'25

x_i: 17,20,23,26,29,32,35; f_i: 6,8,15,17,10,7,4 (3 marks)

$N = 6 + 8 + 15 + 17 + 10 + 7 + 4 = 67$

$\sum f_i x_i = 102 + 160 + 345 + 442 + 290 + 224 + 140 = 1703$

$\bar{x} = \frac{1703}{67} = \boxed{25.42}$

x_i: 5,10,15,20,25,30,35,40; f_i: 2,3,5,6,4,3,2,1 (N-W'25)

$N = 26$, $\sum fx = 10 + 30 + 75 + 120 + 100 + 90 + 70 + 40 = 535$,

$\bar{x} = \frac{535}{26} = \boxed{20.58}$

5.4 Mean Deviation (MD)

Average of absolute deviations from the mean

Mean Deviation — Formulas

Ungrouped Data

$$\text{MD} = \frac{\sum |x_i - \bar{x}|}{n}$$

Grouped / Frequency Data

$$\text{MD} = \frac{\sum f_i |x_i - \bar{x}|}{N} \quad \text{where } N = \sum f_i$$

Steps

1. Compute $\bar{x}$ first 2. Find $|x_i - \bar{x}|$ for each value 3. Multiply by f_i (grouped) 4. Divide by N

MD (Ungrouped) — GTU W'22

Data: 4,6,2,4,5,4,4,5,3,4 (3 marks)

$$n = 10, \;\; \bar{x} = \frac{41}{10} = 4.1$$

x_i	4	6	2	4	5	4	4	5	3	4
$\lvert x_i - 4.1 \rvert$	0.1	1.9	2.1	0.1	0.9	0.1	0.1	0.9	1.1	0.1

$$\sum \lvert x_i - \bar{x} \rvert = 7.4 \qquad \text{MD} = \frac{7.4}{10} = \boxed{0.74}$$

MD (Ungrouped) — GTU Su'25 & N-W'25

30,35,39,41,36,37,43,40,32 — Su'25 (3 marks)

$n = 9$, $\bar{x} = \frac{333}{9} = 37$

$|d_i|$: 7,2,2,4,1,0,6,3,5 $\Rightarrow \sum = 30$ $\quad \text{MD} = \dfrac{30}{9} = \boxed{3.33}$

4,6,8,10,12,14 — N-W'25 (3 marks)

$n = 6$, $\bar{x} = \frac{54}{6} = 9$

$|d_i|$: 5,3,1,1,3,5 $\Rightarrow \sum = 18$ $\quad \text{MD} = \dfrac{18}{6} = \boxed{3}$

MD (Grouped) — GTU Su'24

x_i: 3,4,5,6,7,8; f_i: 1,3,7,5,2,2 (3 marks)

$N = 20$, $\sum fx = 1\cdot3+3\cdot4+7\cdot5+5\cdot6+2\cdot7+2\cdot8 = 3+12+35+30+14+16 = 110$, $\bar{x} = 5.5$

x_i	f_i	$\lvert x_i - 5.5\rvert$	$f\lvert x_i - 5.5\rvert$
3	1	2.5	2.5
4	3	1.5	4.5
5	7	0.5	3.5
6	5	0.5	2.5
7	2	1.5	3.0
8	2	2.5	5.0
20			**21.0**

$$\text{MD} = \frac{21}{20} = \boxed{1.05}$$

Class 30–100, f: 3,7,12,15,8,3,2 (3 marks)

Class	x	f	fx	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
30–40	35	3	105	29	87
40–50	45	7	315	19	133
50–60	55	12	660	9	108
60–70	65	15	975	1	15
70–80	75	8	600	11	88
80–90	85	3	255	21	63
90–100	95	2	190	31	62
		50	**3100**		**556**

$\bar{x} = \frac{3100}{50} = 62$ $\quad$ $\text{MD} = \frac{556}{50} = \boxed{11.12}$

MD (Grouped) — GTU Su'25 & W'23

Class 0–50, f: 5,8,15,16,6 — Su'25 (3 marks)

x: 5,15,25,35,45; $N = 50$; $\sum fx = 25 + 120 + 375 + 560 + 270 = 1350$; $\bar{x} = 27$
$f|x - 27|$: $5 \cdot 22 + 8 \cdot 12 + 15 \cdot 2 + 16 \cdot 8 + 6 \cdot 18 = 110 + 96 + 30 + 128 + 108 = 472$

$$\text{MD} = \frac{472}{50} = \boxed{9.44}$$

x_i: 4,8,11,17,20,24,32; f_i: 3,5,9,5,4,3,1 — W'23/W'24 (3 marks)

$N = 30$; $\sum fx = 12 + 40 + 99 + 85 + 80 + 72 + 32 = 420$; $\bar{x} = 14$
$f|x-14|$: $3\cdot10+5\cdot6+9\cdot3+5\cdot3+4\cdot6+3\cdot10+1\cdot18 = 30+30+27+15+24+30+18 = 174$

$$\text{MD} = \frac{174}{30} = \boxed{5.8}$$

5.5 Standard Deviation (σ)

Root mean square deviation from the mean

Standard Deviation — Formulas

Ungrouped Data

$$\sigma = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}} \quad \text{or} \quad \sigma = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$$

Grouped / Frequency Data

$$\sigma = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}} \quad \text{or} \quad \sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - \bar{x}^2}$$

Variance

$\text{Var} = \sigma^2$

SD (Ungrouped) — GTU W'23

Data: 6,7,10,12,13,4,8,12 (3 marks)

$$n = 8,\ \bar{x} = \frac{72}{8} = 9$$

$$(x_i - 9)^2:\ 9, 4, 1, 9, 16, 25, 1, 9 \quad \sum (x_i - \bar{x})^2 = 74$$

$$\sigma = \sqrt{\frac{74}{8}} = \sqrt{9.25} = \boxed{3.04}$$

Data: 10,12,13,16,20,25 N-W'25 (3 marks)

$$n = 6,\ \bar{x} = 16; \quad (d_i)^2:\ 36, 16, 9, 0, 16, 81; \sum = 158$$

$$\sigma = \sqrt{\frac{158}{6}} = \sqrt{26.33} = \boxed{5.13}$$

SD (Ungrouped) — GTU Su'24 / W'24

120,132,148,136,142,140,165,153 (4 marks)

$n = 8$, $\bar{x} = \dfrac{1136}{8} = 142$

x	$d = x - 142$	d^2
120	-22	484
132	-10	100
148	6	36
136	-6	36
142	0	0
140	-2	4
165	23	529
153	11	121
		1310

$$\sigma = \sqrt{\frac{1310}{8}} = \sqrt{163.75} = \boxed{12.80}$$

SD (Grouped Freq Table) — GTU W'22 / W'24

x_i: 4,8,11,17,20,24,32; f_i: 3,5,9,5,4,3,1 (3 marks)

$N = 30$; $\bar{x} = \frac{420}{30} = 14$

x_i	f_i	$d_i = x_i - 14$	d_i^2	$f_i d_i^2$
4	3	-10	100	300
8	5	-6	36	180
11	9	-3	9	81
17	5	3	9	45
20	4	6	36	144
24	3	10	100	300
32	1	18	324	324
30				**1374**

$$\sigma = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = \boxed{6.77}$$

SD (Grouped Classes) — GTU W'24 (4 marks)

Class 0–100 (width 20), f: 12,38,42,23,5

Class	x	f	fx	$d = x - \bar{x}$	fd^2
0–20	10	12	120	-46	25392
20–40	30	38	1140	-26	25688
40–60	50	42	2100	-6	1512
60–80	70	23	1610	14	4508
80–100	90	5	450	34	5780
		120	**5420**		**62880**

$\bar{x} = \frac{5420}{120} = 45.17 \quad \sigma = \sqrt{\frac{62880}{120}} = \sqrt{524} = \boxed{22.89}$

SD (Grouped Freq Table) — GTU N-W'25

x_i: 2,4,6,8,10,12; f_i: 1,2,3,3,2,1 (3 marks)

$N = 12$; $\bar{x} = \frac{72}{12} = 7$

$(d_i)^2$: 25,9,1,1,9,25; fd^2: 25,18,3,3,18,25; $\sum fd^2 = 92$

$$\sigma = \sqrt{\frac{92}{12}} = \sqrt{7.67} = \boxed{2.77}$$

5.6 Coefficient of Variation (CV)

Relative measure of dispersion

Coefficient of Variation

Formula

$$CV = \frac{\sigma}{\bar{x}} \times 100\%$$

Interpretation

- Lower CV $\Rightarrow$ more consistent / less variable data
- Higher CV $\Rightarrow$ more spread out data
- Used to compare variability of two different datasets

Quick Example

If $\bar{x} = 50$ and $\sigma = 10$, then $CV = \dfrac{10}{50} \times 100 = 20\%$

5.7 GTU PYQ Practice

All exam-style questions from W'22 – N-W'25

PYQ — 3-Mark Questions (Mean & MD)

Try These

1. Find mean: x_i=92,93,97,98,102,104; f_i=3,2,3,2,6,4 *[W'22]*
2. Mean of 15,7,6,a,3 is 7; find a. *[W'22]*
3. MD about mean: 4,6,2,4,5,4,4,5,3,4 *[W'22]*
4. Find MD: x_i=3,4,5,6,7,8; f_i=1,3,7,5,2,2 *[Su'24]*
5. Grouped MD: class 30–100, f=3,7,12,15,8,3,2 *[W'23]*
6. Missing f: mean=19, x_i=6,10,14,18,24,28,30 *[Su'23]*

PYQ — 3-mark (SD) & 4-mark Questions

3-Mark (SD)

1. SD of 6,7,10,12,13,4,8,12 *[W'23]*
2. SD: x_i=4,8,11,17,20,24,32; f_i=3,5,9,5,4,3,1 *[W'22/W'24]*
3. SD of 10,12,13,16,20,25 *[N-W'25]*
4. SD: x_i=2,4,6,8,10,12; f_i=1,2,3,3,2,1 *[N-W'25]*

4-Mark

1. Mean (step deviation): class 21–50, f=8,10,24,30,12,16 *[W'22]*
2. SD of 10,15,7,19,9,21,23,25,26,30 *[Su'23]*
3. SD: 120,132,148,136,142,140,165,153 *[Su'24/W'24]*
4. SD (grouped 0–100): f=12,38,42,23,5 *[W'24]*

Unit 5 — Formula Summary

Central Tendency

Direct: $\bar{x} = \frac{\sum fx}{N}$

Assumed Mean: $\bar{x} = A + \frac{\sum fd}{N}$

Step Dev: $\bar{x} = A + h \cdot \frac{\sum fu}{N}$

Combined: $\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

Dispersion

Range $= x_{\text{max}} - x_{\text{min}}$

MD $= \frac{\sum f|x - \bar{x}|}{N}$

$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{N}}$

CV $= \frac{\sigma}{\bar{x}} \times 100\%$

Exam Weightage (Unit 5)

MCQ (1 mark) $\times$ 2–3 | Short (3 marks) $\times$ 2–3 | Long (4 marks) $\times$ 1

5.8 Correlation

Karl Pearson's Coefficient

Correlation — Definition and Formula

What is Correlation?

Measures the **strength and direction** of linear relationship between two variables x and y.

Karl Pearson's Coefficient r

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \cdot \sum(y-\bar{y})^2}}$$

Shortcut: $r = \dfrac{n\sum xy - \sum x \sum y}{\sqrt{[n\sum x^2 - (\sum x)^2][n\sum y^2 - (\sum y)^2]}}$

Interpretation

$r = +1$: Perfect positive $\quad r = -1$: Perfect negative $\quad r = 0$: No correlation
$|r| > 0.7$: Strong $\quad 0.4 < |r| \leq 0.7$: Moderate $\quad |r| \leq 0.4$: Weak

Correlation Example 1 (GTU W'22, 4-mark)

Find r for: x: 1,2,3,4,5; y: 2,4,5,4,5

x	y	x^2	y^2	xy
1	2	1	4	2
2	4	4	16	8
3	5	9	25	15
4	4	16	16	16
5	5	25	25	25
15	20	55	86	66

Calculation

$$n = 5;\ r = \frac{5(66) - 15 \cdot 20}{\sqrt{[5(55) - 225][5(86) - 400]}} = \frac{330 - 300}{\sqrt{50 \cdot 30}} = \frac{30}{\sqrt{1500}} \approx \mathbf{0.775}$$

Correlation Example 2 (GTU Su'24, W'23)

Su'24: x: 6,2,10,4,8; y: 9,11,5,8,7

$n = 5$; $\sum x = 30$, $\sum y = 40$, $\sum x^2 = 220$, $\sum y^2 = 340$, $\sum xy = 225$

$$r = \frac{5(225) - 30 \cdot 40}{\sqrt{[5(220) - 900][5(340) - 1600]}} = \frac{-75}{\sqrt{200 \cdot 100}} = \frac{-75}{\sqrt{20000}} \approx \mathbf{-0.530}$$

Moderate negative correlation

W'23: x: 1,3,5,7,9; y: 2,6,8,6,10 — check

$n = 5$; $\sum x = 25$, $\sum y = 32$, $\sum x^2 = 165$, $\sum y^2 = 240$, $\sum xy = 190$

$$r = \frac{5(190) - 25 \cdot 32}{\sqrt{[5(165) - 625][5(240) - 1024]}} = \frac{150}{\sqrt{200 \cdot 176}} \approx \mathbf{0.796}$$

Strong positive correlation

5.9 Regression Lines

Lines of Best Fit

Regression — Lines of Best Fit

Two Regression Lines

y **on** x: $y - \bar{y} = b_{yx}(x - \bar{x})$ where $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

x **on** y: $x - \bar{x} = b_{xy}(y - \bar{y})$ where $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$

Shortcut: $b_{yx} = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$

Key Property

Both regression lines pass through $(\bar{x}, \bar{y})$

$r^2 = b_{yx} \cdot b_{xy} \Rightarrow r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

Regression Example (GTU W'24, 4-mark)

x: 2,4,6,8,10; y: 5,7,9,8,11 — find both regression lines

$n = 5$; $\sum x = 30$, $\sum y = 40$, $\sum x^2 = 220$, $\sum y^2 = 340$, $\sum xy = 258$
$\bar{x} = 6$, $\bar{y} = 8$

y on x

$$b_{yx} = \frac{5(258) - 30 \cdot 40}{5(220) - 900} = \frac{90}{200} = 0.45$$

$y - 8 = 0.45(x - 6) \Rightarrow y = 0.45x + 5.3$

x on y

$$b_{xy} = \frac{5(258) - 30 \cdot 40}{5(340) - 1600} = \frac{90}{100} = 0.9$$

$x - 6 = 0.9(y - 8) \Rightarrow x = 0.9y - 1.2$

Regression: Finding r from Regression Lines (GTU W'23)

W'23: given $4x - 5y + 33 = 0$ and $20x - 9y - 107 = 0$

Identify which is y on x (steeper slope $= b_{yx}$):

From $4x - 5y + 33 = 0$: $y = \frac{4}{5}x + \frac{33}{5} \Rightarrow b_{yx} = 0.8$

From $20x - 9y - 107 = 0$: $x = \frac{9}{20}y + \frac{107}{20} \Rightarrow b_{xy} = \frac{9}{20} = 0.45$

$r = \sqrt{0.8 \times 0.45} = \sqrt{0.36} = \mathbf{0.6}$

Mean: solve simultaneously: $\bar{x} = 13$, $\bar{y} = 17$

PYQ — Correlation (4-mark practice set)

All PYQ Correlation Questions

1. x: 2,4,6,8,10; y: 5,7,9,8,11 — find r [Su'23]
2. x: 1,2,3,4,5,6; y: 6,4,3,5,4,2 — find r [W'22,W'24]
3. x: 9,8,7,6,5; y: 10,9,8,7,11 — r? [Su'25]
4. x: 12,10,8,6,4; y: 20,22,24,26,28 — r? [N-W'25]
5. Given $b_{yx} = 0.6$, $b_{xy} = 0.4$: find r [MCQ type]

PYQ Regression

1. Find both regression lines for x: 1,2,3,4,5; y: 3,4,4,5,4 [W'22]
2. Lines: $3x - 4y + 8 = 0$; $4x - 3y - 1 = 0$; find $\bar{x}, \bar{y}, r$ [Su'24]

SD — Shortcut Formula Examples

Shortcut SD Formula

$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$ (avoid big decimals!)

GTU Su'25: x: 3,8,13,18,23; f: 7,10,15,10,8 ($N = 50$)

$\sum fx = 3(7) + 8(10) + 13(15) + 18(10) + 23(8) = 21 + 80 + 195 + 180 + 184 = 660$
$\sum fx^2 = 63 + 640 + 2535 + 3240 + 4232 = 10710$
$\bar{x} = 660/50 = 13.2$; $\sigma = \sqrt{10710/50 - 13.2^2} = \sqrt{214.2 - 174.24} = \sqrt{39.96} \approx 6.32$

Combined SD & Two-Dataset Comparison (GTU N-W'25)

Which batch is more consistent?

Batch A: $\bar{x}_A = 40$, $\sigma_A = 6$ Batch B: $\bar{x}_B = 50$, $\sigma_B = 7$
$\text{CV}_A = \frac{6}{40} \times 100 = 15\%$ $\text{CV}_B = \frac{7}{50} \times 100 = 14\%$
$\text{CV}_B < \text{CV}_A \Rightarrow$ Batch B is **more consistent**

GTU W'24: Which dataset more variable?

Dataset 1: mean=25, $\sigma = 4$; $\text{CV}_1 = 16\%$
Dataset 2: mean=12, $\sigma = 3$; $\text{CV}_2 = 25\%$
Dataset 2 is **more variable**

Weighted Mean & Median (Quick Coverage)

Weighted Mean

$$\bar{x}_w = \frac{\sum w_i x_i}{\sum w_i}$$

Example: marks 70,80,60 with weights 3,4,2:

$\bar{x}_w = \frac{3(70)+4(80)+2(60)}{9} = \frac{210+320+120}{9} = \frac{650}{9} \approx 72.2$

Median (Grouped)

$$\text{Median} = L + \frac{\frac{N}{2} - F}{f} \times h$$

L=lower boundary of median class; F=cumulative freq before; f=freq of median class; h=class width

1-Mark Questions

1. If CV of A$= 12\%$, B$= 18\%$, which is more consistent? **A**
2. SD of 5,5,5,5,5 is ______. **0**
3. If all values multiplied by k, new SD $=$ ______. $k\sigma$
4. $r =?$ if $b_{yx} = 0.8$ and $b_{xy} = 0.5$. $\sqrt{0.4} \approx 0.632$
5. Regression y on x: passes through $(\bar{x}, \bar{y})$? **Yes**
6. Range of r: ______. $[-1, 1]$
7. Mean deviation is least about ______. **median**
8. σ^2 is called ______. **variance**

PYQ 4-mark — Full Correlation & Regression Set

Su'24: Find regression line y on x

x: 5,10,15,20,25; y: 16,19,23,26,30; $n = 5$
$\sum x = 75$, $\sum y = 114$, $\sum x^2 = 1375$, $\sum xy = 1890$
$b_{yx} = \frac{5(1890)-75\cdot 114}{5(1375)-5625} = \frac{810}{1250} = 0.648$
$\bar{x} = 15$, $\bar{y} = 22.8$; Line: $y - 22.8 = 0.648(x - 15) \Rightarrow y = 0.648x + 13.08$

N-W'25: Find r from: x: 10,20,30,40,50; y: 40,30,20,10,5

$\sum x = 150$, $\sum y = 105$, $\sum x^2 = 5500$, $\sum y^2 = 3125$, $\sum xy = 2750$
$r = \frac{5(2750)-150\cdot 105}{\sqrt{[5(5500)-22500][5(3125)-11025]}} = \frac{-2250}{\sqrt{5000\cdot 4600}} \approx -0.936$ (strong negative)

Standard Deviation — Step Deviation Method

Formula for Grouped Data (Step Deviation)

$$\sigma = h\sqrt{\frac{\sum fu^2}{N} - \left(\frac{\sum fu}{N}\right)^2} \text{ where } u = \frac{x-A}{h}$$

GTU Su'25 (4-mark): Class 0–50, f: 5,8,15,12,10 ($N = 50$, $A = 25$, $h = 10$)

Class	x	u	fu	fu^2
0–10	5	-2	-10	20
10–20	15	-1	-8	8
20–30	25	0	0	0
30–40	35	1	12	12
40–50	45	2	20	40
			14	80

$\sigma = 10\sqrt{80/50 - (14/50)^2} = 10\sqrt{1.6 - 0.0784} = 10\sqrt{1.5216} \approx \mathbf{12.34}$

Frequency Distribution — Missing Frequency Problems

GTU W'24: Mean = 50, find missing f_3

x:	10	30	50	70	90
f:	17	f_2	32	f_4	19

Given: $N = 120$, mean $= 50$; $f_2 + f_4 = 52$; $\sum fx = \text{mean} \times N = 6000$

$170 + 30f_2 + 1600 + 70f_4 + 1710 = 6000 \Rightarrow 30f_2 + 70f_4 = 2520$

Solve with $f_2 + f_4 = 52$: $f_2 = 28$, $f_4 = 24$

Revision Flash Sheet — Unit 5

All Key Formulas at a Glance

Mean (direct)	$\bar{x} = \frac{\sum fx}{N}$
Mean (assumed)	$\bar{x} = A + \frac{\sum fd}{N}$
Mean (step dev)	$\bar{x} = A + h \cdot \frac{\sum fu}{N}$
MD	$\frac{\sum f\lvert x-\bar{x}\rvert}{N}$
SD (direct)	$\sqrt{\frac{\sum f(x-\bar{x})^2}{N}}$
SD (shortcut)	$\sqrt{\frac{\sum fx^2}{N} - \bar{x}^2}$
SD (step dev)	$h\sqrt{\frac{\sum fu^2}{N} - \left(\frac{\sum fu}{N}\right)^2}$
CV	$\frac{\sigma}{\bar{x}} \times 100\%$
Pearson r	$\frac{n\sum xy-\sum x\sum y}{\sqrt{[\ldots][\ldots]}}$
b_{yx}	$\frac{n\sum xy-\sum x\sum y}{n\sum x^2-(\sum x)^2}$
r from b's	$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

GTU Frequency Guide — Unit 5 All Topics

Marks Distribution Across Papers

Topic	Marks	Frequency
Mean (direct/step dev)	3–4	Every paper
Mean Deviation	3	1–2 per paper
Standard Deviation	3–4	Every paper
CV comparison	3	W'24, Su'25, N-W'25
Missing frequency	3–4	W'22, Su'23, W'24
Correlation (Pearson r)	4	W'22, Su'24, W'23, W'24
Regression lines	4	Su'24, W'23, N-W'25
MCQs (all formulae)	1 each	2–3 per paper

Unit 5 & Course Complete!

Applied Mathematics — DI02000011

All 5 units covered. Best of luck!

SD Properties — Important Results

Effect of Change on SD

Add constant k	SD unchanged (σ stays same)
Multiply by k	New SD $= k\sigma$
All values equal	SD $= 0$
First n **naturals**	$\sigma = \sqrt{\frac{n^2-1}{12}}$

Variance vs SD

Variance $= \sigma^2$; SD $= \sigma = \sqrt{\text{Variance}}$

Variance of first 6 naturals $= \frac{6^2-1}{12} = \frac{35}{12} \approx 2.917$; SD$\approx 1.708$

Mixed PYQ Set — 1-mark & 3-mark (All Years)

3-Mark: Quick Calculations

1. Find mean: 15,20,25,30,35 — $\bar{x} = 25$
2. SD of 2,4,4,4,5,5,7,9 — $\sigma = 2$ $[W'22]$
3. If $\bar{x} = 30$, $\sigma = 6$: find CV — CV $= 20\%$
4. $r = 0.8$, $\sigma_x = 3$, $\sigma_y = 4$: find b_{yx} — $b_{yx} = \frac{0.8\times4}{3} \approx 1.067$
5. MD about mean: 4,7,2,9,3 — $\bar{x} = 5$; MD$= 2.4$

Spearman's Rank Correlation

$$r_s = 1 - \frac{6\sum d_i^2}{n(n^2-1)} \text{ where } d_i = R(x_i) - R(y_i)$$

N-W'25: x: 56,75,45,71,62,64,58,80,76; y: 66,70,40,60,65,56,59,77,67

Rank x: 2,7,1,6,4,5,3,9,8 Rank y: 5,8,1,3,4,2,7,9,6
d: $-3, -1, 0, 3, 0, 3, -4, 0, 2$; $\sum d^2 = 9+1+0+9+0+9+16+0+4 = 48$
$r_s = 1 - \frac{6(48)}{9(80)} = 1 - \frac{288}{720} = 1 - 0.4 = \mathbf{0.6}$

Exam Strategy — Unit 5

How to Score Full Marks in Statistics

- **MCQ:** Know $r \in [-1, 1]$; lower CV = more consistent; SD of constant data = 0
- **3-mark:** Show table (x, f, fx, etc.) — don't skip steps
- **4-mark:** For correlation, show all five columns (x, y, x^2, y^2, xy) + totals
- **Regression:** compute both b_{yx} & b_{xy} even if only one line asked
- **SD:** Use shortcut formula for grouped data to save time

Time Allocation (Unit 5 in 80-mark paper)

MCQ: 5 min | Short: 10 min each | Long: 15 min