Maths (4320002) - Problem Solver

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Contents

CHAPTER 1

Matrices

1.1 Construction of a Matrix

Methodology:

Constructing a Matrix from a Formula Sometimes a matrix must be constructed based on a specific mathematical formula for its individual elements a_{ij}.

1. Identify the order of the matrix $m \times n$. This determines the number of rows m and columns n.
2. Write down the generic matrix with elements $a_{11}, a_{12}, \ldots, a_{mn}$.
3. Substitute the row index i and column index j into the given formula to compute the exact value of each element.

Worked Example:

Construct a 2x2 Matrix Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by the formula $a_{ij} = \frac{(i+j)^2}{2}$.

Solution:

1. The matrix is 2×2, so it has the general form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

2. Compute each element using $a_{ij} = \frac{(i+j)^2}{2}$:

$$\begin{aligned} a_{11} &= \frac{(1+1)^2}{2} = \frac{4}{2} = 2 \\ a_{12} &= \frac{(1+2)^2}{2} = \frac{9}{2} = 4.5 \\ a_{21} &= \frac{(2+1)^2}{2} = \frac{9}{2} = 4.5 \\ a_{22} &= \frac{(2+2)^2}{2} = \frac{16}{2} = 8 \end{aligned}$$

3. Substitute these values back into the matrix:

$$A = \begin{bmatrix} 2 & 4.5 \\ 4.5 & 8 \end{bmatrix}$$

1.2 Matrix Operations

Methodology:

Matrix Addition and Subtraction To add or subtract two matrices they must have the exact same order (identical number of rows and columns).

1. Verify both matrices A and B have order $m \times n$.
2. Add or subtract the corresponding elements: $C_{ij} = A_{ij} \pm B_{ij}$.

Worked Example:

Add and Subtract Two Matrices Given $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix}$. Find $A + B$ and $A - B$.

Solution:

1. Both matrices are 2×2. The operations are valid.
2. Addition:
$$A + B = \begin{bmatrix} 2+5 & 4+1 \\ 1+0 & 3+(-2) \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 1 & 1 \end{bmatrix}$$
3. Subtraction:
$$A - B = \begin{bmatrix} 2-5 & 4-1 \\ 1-0 & 3-(-2) \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & 5 \end{bmatrix}$$

Methodology:

Matrix Multiplication To multiply two matrices A and B (to form AB) the number of columns in A must equal the number of rows in B.

1. Check orders: If A is $m \times p$ and B is $p \times n$, then the resulting matrix AB will be of order $m \times n$.
2. Multiply rows of A by columns of B: The element in the i-th row and j-th column of AB is the dot product of the i-th row of A and the j-th column of B.

Worked Example:

Multiply Two Matrices Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$. Find AB.

Solution:

1. Order of A is 2×2 and B is 2×2. The product AB is defined and will be 2×2.
2. Compute the elements by multiplying rows by columns:
$$\begin{aligned} AB &= \begin{bmatrix} (1)(2)+(2)(1) & (1)(0)+(2)(5) \\ (3)(2)+(4)(1) & (3)(0)+(4)(5) \end{bmatrix} \\ &= \begin{bmatrix} 2+2 & 0+10 \\ 6+4 & 0+20 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 10 \\ 10 & 20 \end{bmatrix} \end{aligned}$$

1.3 Transpose and Adjoint

Methodology:

Transpose of a Matrix The transpose of a matrix A denoted as A^T or A' is found by interchanging its rows and columns.

1. Make the first row of A the first column of A^T.
2. Make the second row of A the second column of A^T and so on for all rows.

Methodology:

Adjoint of a Matrix The adjoint of a square matrix A denoted by $\text{adj}(A)$ is the transpose of the cofactor matrix.

1. Find the minor M_{ij} for each element a_{ij} by deleting the i-th row and j-th column and finding the determinant of the remaining sub-matrix.
2. Find the cofactor $C_{ij} = (-1)^{i+j}M_{ij}$.
3. Form the cofactor matrix C.
4. Transpose the cofactor matrix to get the adjoint: $\text{adj}(A) = C^T$.

Worked Example:

Find the Adjoint of a 3x3 Matrix Find the adjoint of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.

Solution:

1. Find the cofactors $C_{ij} = (-1)^{i+j}M_{ij}$:

$$C_{11} = +\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} = (0 - 24) = -24$$

$$C_{12} = -\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} = -(0 - 20) = 20$$

$$C_{13} = +\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} = (0 - 5) = -5$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} = -(0 - 18) = 18$$

$$C_{22} = +\begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = (0 - 15) = -15$$

$$C_{23} = -\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = -(6 - 10) = 4$$

$$C_{31} = +\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = (8 - 3) = 5$$

$$C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4 - 0) = -4$$

$$C_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (1 - 0) = 1$$

2. Construct the cofactor matrix C:
$$C = \begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$$

3. Transpose C to find the adjoint:
$$\text{adj}(A) = C^T = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

1.4 Inverse of a Matrix

Methodology:

Inverse using Adjoint Method The inverse of a square matrix A is denoted as A^{-1}. It exists only if A is non-singular ($|A| \neq 0$).

1. Calculate the determinant of the matrix $|A|$. If $|A| = 0$ the inverse does not exist.
2. Find the adjoint of the matrix $\text{adj}(A)$.
3. Multiply the adjoint by $\frac{1}{|A|}$ to find the inverse:
$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Worked Example:

Find the Inverse of a 2x2 Matrix Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$.

Solution:

1. Find the determinant $|A|$:
$$|A| = (4)(2) - (3)(2) = 8 - 6 = 2$$
Since $|A| \neq 0$, A^{-1} exists.
2. Find the adjoint of A. For a 2×2 matrix swap elements on the main diagonal and change the signs of the off-diagonal elements:
$$\text{adj}(A) = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$
3. Compute the inverse:
$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1.5 \\ -1 & 2 \end{bmatrix}$$

1.5 Solution of Simultaneous Linear Equations

Methodology:

Matrix Inversion Method A system of linear equations can be represented as $AX = B$.

1. Write the equations in matrix form: Let A be the coefficient matrix X be the column matrix of variables and B be the column matrix of constants.
2. Find the determinant $|A|$. If $|A| = 0$ the system has no unique solution.

3. Find A^{-1} using the formula $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

4. Multiply A^{-1} by B to find X: $X = A^{-1}B$.

Worked Example:

Solve a System of Equations Solve the following system using the matrix inversion method:

$$\begin{aligned} 3x + 2y &= 7 \\ 4x - y &= 2 \end{aligned}$$

Solution:

1. Write in matrix form $AX = B$:
$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$
Here $A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

2. Find the determinant $|A|$:
$$|A| = (3)(-1) - (2)(4) = -3 - 8 = -11$$

3. Find adj(A) by swapping the main diagonal and negating the off-diagonal:
$$\text{adj}(A) = \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix}$$

4. Compute A^{-1}:
$$A^{-1} = \frac{1}{-11} \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix}$$

5. Multiply A^{-1} by B to find X:
$$\begin{aligned} X = A^{-1}B &= \frac{-1}{11} \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} \\ &= \frac{-1}{11} \begin{bmatrix} (-1)(7) + (-2)(2) \\ (-4)(7) + (3)(2) \end{bmatrix} \\ &= \frac{-1}{11} \begin{bmatrix} -7 - 4 \\ -28 + 6 \end{bmatrix} \\ &= \frac{-1}{11} \begin{bmatrix} -11 \\ -22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

6. Therefore the solution is $x = 1$ and $y = 2$.

CHAPTER 2

Unit 1: Determinants and Matrices

2.1 Matrix Fundamentals: Order of a Matrix

Methodology:

How to determine the order of a matrix and its products

1. The order of a matrix is written as rows $\times$ columns. Always count the horizontal rows first, and then the vertical columns.
2. When multiplying two matrices, say A and B:
   - Let the order of matrix A be $m \times n$.
   - Let the order of matrix B be $p \times q$.
   - Multiplication (AB) is only possible if $n = p$ (columns of A must equal rows of B).
   - The order of the resulting matrix AB will be $m \times q$ (rows of A $\times$ columns of B).

Worked Example:

Finding Matrix Order and Multiplication Order **Problem:** 1. Find the order of the matrix $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 0 \end{bmatrix}$.
2. If A is a matrix of order 2×3 and B is of order 3×2, what is the order of AB?
Solution:
Part 1: The given matrix has 2 horizontal rows and 3 vertical columns. Therefore, the order is 2×3.
Part 2: Order of $A = 2 \times 3$
Order of $B = 3 \times 2$
Since the columns of A (which is 3) match the rows of B (which is 3), they can be multiplied. The order of the resulting matrix AB is formed by taking the rows of A and columns of B:
Order of $AB = 2 \times 2$.

2.2 Matrix Equations (Addition and Scalar Multiplication)

Methodology:

How to solve Matrix Equations (e.g., $aA + X = bB$)

1. Isolate the unknown matrix X on one side of the equation (e.g., move other matrices so $X = bB - aA$).
2. Substitute the given matrices A and B into the rearranged equation.
3. Perform scalar multiplication by multiplying every single element inside matrix A by the scalar number a, and every element in B by b.
4. Add or subtract the corresponding elements of the resulting matrices to find the final elements of X.

Worked Example:

Finding an unknown matrix in an equation **Problem:** If $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, find the matrix X such that $2A + X = 3B$.

Solution:

Step 1: Isolate X.

Given $2A + X = 3B$, rearrange to find X:

$$X = 3B - 2A$$

Step 2: Substitute A and B.

$$X = 3\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

Step 3: Multiply by scalars.

$$X = \begin{bmatrix} 3(3) & 3(2) \\ 3(1) & 3(4) \end{bmatrix} - \begin{bmatrix} 2(2) & 2(-1) \\ 2(4) & 2(3) \end{bmatrix}$$

$$X = \begin{bmatrix} 9 & 6 \\ 3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 8 & 6 \end{bmatrix}$$

Step 4: Subtract corresponding elements.

$$X = \begin{bmatrix} 9-4 & 6-(-2) \\ 3-8 & 12-6 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & 8 \\ -5 & 6 \end{bmatrix}$$

2.3 Matrix Multiplication

Methodology:

How to multiply two matrices ($A \times B$)

1. Verify that multiplication is possible (columns of first matrix = rows of second matrix).
2. To find the element in the i-th row and j-th column of the product matrix, take the entire i-th row of the first matrix and the entire j-th column of the second matrix.
3. Multiply their corresponding elements sequentially and add the results together.
4. Repeat this process for every row-column combination to build the complete new matrix.

Worked Example:

Matrix Multiplication and Transpose **Problem:** If $A = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, find $(AB)^T$.

Solution:

Step 1: Find the product AB

Both matrices are of order 2×2, so their product will also be a 2×2 matrix.

$$AB = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

Multiply rows of A by columns of B:

$$AB = \begin{bmatrix} (5)(1) + (4)(2) & (5)(3) + (4)(1) \\ (4)(1) + (3)(2) & (4)(3) + (3)(1) \end{bmatrix}$$

Simplify the terms:

$$AB = \begin{bmatrix} 5+8 & 15+4 \\ 4+6 & 12+3 \end{bmatrix} = \begin{bmatrix} 13 & 19 \\ 10 & 15 \end{bmatrix}$$

Step 2: Find the transpose $(AB)^T$
To find the transpose, swap the rows and columns of the matrix AB. (Row 1 becomes Column 1, etc.)

$$(AB)^T = \begin{bmatrix} 13 & 10 \\ 19 & 15 \end{bmatrix}$$

2.4 Transpose of a Matrix and its Properties

Methodology:

How to find the Transpose and verify matrix properties

1. **Finding Transpose (A^T):** Swap the rows and columns of the matrix. The 1st row becomes the 1st column, the 2nd row becomes the 2nd column, etc.
2. **Verifying an equation (e.g., $(A+B)^T = A^T + B^T$):**
   - **Solve LHS:** First, perform the operation inside the parentheses (e.g., add A and B). Then, take the transpose of the final result. Label this as Equation (1).
   - **Solve RHS:** First, find the transpose of the individual matrices (A^T and B^T). Then, perform the required operation (e.g., add them together). Label this as Equation (2).
   - Compare Equation (1) and Equation (2). If they are completely identical, the property is verified.

Worked Example:

Verifying the Transpose Addition Property **Problem:** If $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$, prove that $(A+B)^T = A^T + B^T$.
Solution:
Step 1: Calculate LHS: $(A+B)^T$
First, find $A + B$:

$$A + B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ 6 & 8 & 1 \end{bmatrix}$$

Now, take the transpose by turning rows into columns:

$$(A+B)^T = \begin{bmatrix} 3 & 6 \\ 5 & 8 \\ 3 & 1 \end{bmatrix} \quad \text{— (Equation 1)}$$

Step 2: Calculate RHS: $A^T + B^T$
First, find A^T and B^T individually:

$$A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Now, add A^T and B^T:

$$A^T + B^T = \begin{bmatrix} 2+1 & 4+2 \\ 3+2 & 5+3 \\ -1+4 & 0+1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & 8 \\ 3 & 1 \end{bmatrix} \quad \text{— (Equation 2)}$$

Step 3: Compare LHS and RHS
From Equation (1) and Equation (2), both matrices are equal.
Therefore, $(A+B)^T = A^T + B^T$ is proved.

2.5 Adjoint and Inverse of a Matrix (A^{-1})

Methodology:

How to find the Inverse of a 3×3 or 2×2 Matrix

1. **Find the Determinant $|A|$:** If $|A| = 0$, stop immediately—the inverse does not exist (the matrix is singular).
2. **Find the Cofactor Matrix:** For each element a_{ij}, find its cofactor by hiding its row and column, calculating the determinant of the remaining elements, and multiplying by $(-1)^{i+j}$ (which assigns an alternating $+/-$ sign pattern).
3. **Find the Adjoint (adj A):** Take the transpose of the Cofactor Matrix.
   - **Shortcut for 2×2 Adjoint:** For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, swap elements on the main diagonal (a and d) and flip the signs of the other two elements (b and c) to get $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
4. **Calculate the Inverse:** Use the formula $A^{-1} = \frac{1}{|A|}\text{adj } A$. Multiply every term in the adjoint matrix by the scalar $\frac{1}{|A|}$.

Worked Example:

Finding the Inverse of a 3×3 Matrix **Problem:** If $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$, find A^{-1}.

Solution:
Step 1: Find Determinant $|A|$

$$|A| = 2\begin{vmatrix} 0 & 4 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$|A| = 2(0(1) - 4(-1)) + 1(1(1) - 4(1)) + 0$$

$$|A| = 2(4) + 1(-3) = 8 - 3 = 5$$

Since $|A| \neq 0$, the inverse exists.
Step 2: Find the Cofactor Matrix
Calculate cofactors for each position using $C_{ij} = (-1)^{i+j} M_{ij}$:

- $C_{11} = +(0 - (-4)) = 4$
- $C_{12} = -(1 - 4) = 3$
- $C_{13} = +(-1 - 0) = -1$

- $C_{21} = -(-1 - 0) = 1$
- $C_{22} = +(2 - 0) = 2$
- $C_{23} = -(-2 - (-1)) = 1$
- $C_{31} = +(-4 - 0) = -4$
- $C_{32} = -(8 - 0) = -8$
- $C_{33} = +(0 - (-1)) = 1$

Cofactor Matrix $= \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 1 \\ -4 & -8 & 1 \end{bmatrix}$

Step 3: Find Adjoint (adj A)

Take the transpose of the cofactor matrix:

$$\text{adj } A = \begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$$

Step 4: Find A^{-1}

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$$

2.6 Solving Linear Equations using Matrix Method

Methodology:

How to solve a system of linear equations

1. **Formulate Matrices:** Write the given system of equations in the standard matrix form $AX = B$.
   - A: Matrix of coefficients from the equations.
   - X: Column matrix of the variables (e.g., $\begin{bmatrix} x \\ y \end{bmatrix}$).
   - B: Column matrix of constants on the right side of the equations.
2. **Check Determinant:** Find $|A|$. The equations can be solved only if $|A| \neq 0$.
3. **Find Inverse:** Calculate $A^{-1} = \frac{1}{|A|}\text{adj } A$.
4. **Solve for X:** Multiply A^{-1} by B using the formula $X = A^{-1}B$.
5. Equate the resulting matrix elements to the variables in matrix X to get final answers.

Worked Example:

Solving 2-variable linear equations **Problem:** Solve the equations $3x - y = 1$ and $x + 2y = 5$ by matrix method.

Solution:

Step 1: Write in matrix form $AX = B$.

$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Step 2: Find $|A|$.

$$|A| = (3)(2) - (-1)(1) = 6 + 1 = 7$$

Since $|A| \neq 0$, a unique solution exists.

Step 3: Find A^{-1}.

Using the 2×2 shortcut, swap diagonal elements and flip signs of the others to find adj A:

$$\text{adj } A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Step 4: Find $X = A^{-1}B$.

$$X = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Multiply the matrices:

$$X = \frac{1}{7}\begin{bmatrix} (2)(1) + (1)(5) \\ (-1)(1) + (3)(5) \end{bmatrix}$$

$$X = \frac{1}{7}\begin{bmatrix} 2 + 5 \\ -1 + 15 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

Multiply by the scalar $\frac{1}{7}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Step 5: Final Answer.

By equating corresponding elements, we get:

$$x = 1$$

$$y = 2$$

CHAPTER 3

Differentiation and its Applications

Differentiation is a core mathematical tool used to determine the rate at which quantities change. This chapter presents systematic methods for differentiating various classes of functions and demonstrates their applications in geometry and optimization.

3.1 Derivatives of Standard Functions

Methodology:

Standard Derivative Formulas The following standard formulas are the fundamental building blocks for differentiating complex functions.

Function $f(x)$	**Derivative** $f'(x)$
c (constant)	0
x^n	nx^{n-1}
e^x	e^x
a^x $(a > 0)$	$a^x \log_e a$
$\log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Worked Example:

Find the derivative of $y = x^5 + 3\sin x - 4e^x + 7$ **Step 1: Apply the sum and difference rule.**
The derivative of a sum or difference is the sum or difference of the derivatives. Constant multiples can be factored out.

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) + 3\frac{d}{dx}(\sin x) - 4\frac{d}{dx}(e^x) + \frac{d}{dx}(7)$$

Step 2: Apply standard formulas.
Substitute the standard derivative for each term:

$$\frac{dy}{dx} = 5x^4 + 3(\cos x) - 4(e^x) + 0$$

$$\frac{dy}{dx} = 5x^4 + 3\cos x - 4e^x$$

3.2 Rules of Differentiation

Methodology:

Product and Quotient Rules When functions are multiplied or divided specific rules must be applied. Let u and v be differentiable functions of x.

1. Product Rule: To differentiate the product of two functions $y = u \cdot v$:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

2. Quotient Rule: To differentiate the quotient of two functions $y = \frac{u}{v}$ (where $v \neq 0$):

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked Example:

Find the derivative of $y = x^2 \sin x$ **Step 1: Identify functions for the Product Rule.**
Let the first function be $u = x^2$ and the second function be $v = \sin x$.
Step 2: Apply the Product Rule formula.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2)$$

Step 3: Differentiate the components and simplify.

$$\frac{dy}{dx} = x^2(\cos x) + (\sin x)(2x)$$

$$\frac{dy}{dx} = x^2\cos x + 2x\sin x$$

Worked Example:

Find the derivative of $y = \frac{e^x}{x^3}$ **Step 1: Identify functions for the Quotient Rule.**
Let the numerator be $u = e^x$ and the denominator be $v = x^3$.
Step 2: Apply the Quotient Rule formula.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x^3\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x^3)}{(x^3)^2}$$

Step 3: Differentiate the components and simplify.

$$\frac{dy}{dx} = \frac{x^3(e^x) - e^x(3x^2)}{x^6}$$

Factor out common terms in the numerator to simplify:

$$\frac{dy}{dx} = \frac{x^2e^x(x-3)}{x^6} = \frac{e^x(x-3)}{x^4}$$

3.3 Derivative of Composite Functions

Methodology:

Chain Rule Algorithm The Chain Rule is utilized to differentiate composite functions of the form $y = f(g(x))$. **Step 1:** Identify the outer function $f(u)$ and the inner function $u = g(x)$. **Step 2:** Differentiate the outer function with respect to u to get $\frac{dy}{du}$. **Step 3:** Differentiate the inner function with respect to x to get $\frac{du}{dx}$. **Step 4:** Multiply the results using the formula:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Worked Example:

Find the derivative of $y = \sin(x^2 + 5)$ **Step 1: Identify outer and inner functions.**
Let the inner function be $u = x^2 + 5$. Then the outer function is $y = \sin u$.
Step 2: Differentiate each part separately.
Derivative of the outer function with respect to u:

$$\frac{dy}{du} = \frac{d}{du}(\sin u) = \cos u$$

Derivative of the inner function with respect to x:

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 5) = 2x$$

Step 3: Apply the Chain Rule.
Multiply the derivatives:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u) \cdot (2x)$$

Step 4: Substitute u back to express the final answer in terms of x.

$$\frac{dy}{dx} = 2x \cos(x^2 + 5)$$

3.4 Derivative of Implicit Functions

Methodology:

Implicit Differentiation Method When an equation relates x and y implicitly and cannot be easily rearranged to the form $y = f(x)$ explicitly apply implicit differentiation.
Step 1: Differentiate both sides of the given equation with respect to x. **Step 2:** Treat y as a differentiable function of x. Whenever a term containing y is differentiated apply the chain rule by multiplying the result by $\frac{dy}{dx}$. **Step 3:** Collect all terms containing $\frac{dy}{dx}$ on one side of the equation and move all remaining terms to the opposite side. **Step 4:** Factor out $\frac{dy}{dx}$ and divide to isolate and solve for it.

Worked Example:

Find the derivative if $x^3 + y^3 = 3axy$ **Step 1: Differentiate both sides with respect to x.**

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3axy)$$

Step 2: Apply derivative rules.
Use the chain rule for y^3 and the product rule for xy:

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y \cdot 1\right)$$

Step 3: Expand and collect terms with $\frac{dy}{dx}$.
Divide the entire equation by 3 to simplify:

$$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

Bring all $\frac{dy}{dx}$ terms to the left side:

$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$

Step 4: Solve for $\frac{dy}{dx}$.
Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - ax) = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

3.5 Derivative of Parametric Functions

Methodology:

Parametric Differentiation Formula If variables x and y are both expressed as functions of a third variable known as a parameter (e.g. t or θ) their derivative is found by differentiating both equations with respect to the parameter.
Step 1: Differentiate x with respect to the parameter to find $\frac{dx}{dt}$ or $\frac{dx}{d\theta}$. **Step 2:** Differentiate y with respect to the parameter to find $\frac{dy}{dt}$ or $\frac{dy}{d\theta}$. **Step 3:** Divide the derivative of y by the derivative of x to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

Worked Example:

Find the derivative if $x = a\cos\theta$ and $y = b\sin\theta$ **Step 1: Differentiate x with respect to the parameter θ.**

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = -a\sin\theta$$

Step 2: Differentiate y with respect to the parameter θ.

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin\theta) = b\cos\theta$$

Step 3: Divide to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

3.6 Higher Order Derivatives

Methodology:

Finding Second Order Derivatives The second order derivative is simply the derivative of the first derivative. It measures the rate of change of the rate of change.
Step 1: Differentiate the function $y = f(x)$ to find the first derivative $y' = \frac{dy}{dx}$. **Step 2:** Differentiate the resulting expression again with respect to x to find the second derivative.

$$\text{Notation: } y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Worked Example:

Find the second order derivative of $y = x^4 - 5x^2 + \sin x$ **Step 1: Find the first derivative.** Differentiate each term with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - \frac{d}{dx}(5x^2) + \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = 4x^3 - 10x + \cos x$$

Step 2: Differentiate the first derivative to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3) - \frac{d}{dx}(10x) + \frac{d}{dx}(\cos x)$$
$$\frac{d^2y}{dx^2} = 12x^2 - 10 - \sin x$$

3.7 Applications of Derivatives

3.7.1 Equation of Tangent and Normal

Methodology:

Equations of Tangent and Normal Lines The first derivative of a curve $y = f(x)$ evaluated at a specific point gives the slope of the tangent line to the curve at that point.
Step 1: Find the derivative $\frac{dy}{dx}$. **Step 2:** Substitute the given coordinates (x_1, y_1) into the derivative to compute the slope of the tangent line $m = \frac{dy}{dx}\Big|_{(x_1,y_1)}$. **Step 3:** Determine the slope of the normal line. Since the normal is perpendicular to the tangent its slope is $m' = -\frac{1}{m}$. **Step 4:** Use the point-slope form of a linear equation to write the final equations:

- **Tangent Line Equation:** $y - y_1 = m(x - x_1)$
- **Normal Line Equation:** $y - y_1 = -\frac{1}{m}(x - x_1)$

Worked Example:

Find equations for tangent and normal to $y = x^2 - 4x + 3$ at $x = 3$ and $y = 0$ **Step 1: Find the derivative of the curve.**

$$y = x^2 - 4x + 3$$
$$\frac{dy}{dx} = 2x - 4$$

Step 2: Evaluate the slope of the tangent line at the point $(3, 0)$.
Substitute $x = 3$ into the derivative:

$$m = \left.\frac{dy}{dx}\right|_{x=3} = 2(3) - 4 = 6 - 4 = 2$$

Step 3: Form the equation of the tangent line.
Using the point-slope form with slope $m = 2$ and point $(3, 0)$:

$$y - 0 = 2(x - 3)$$

$$y = 2x - 6$$

$$2x - y - 6 = 0$$

Step 4: Form the equation of the normal line.
The slope of the normal line is the negative reciprocal of the tangent slope $m' = -\frac{1}{2}$.

$$y - 0 = -\frac{1}{2}(x - 3)$$

Multiply by 2:

$$2y = -x + 3$$

$$x + 2y - 3 = 0$$

3.7.2 Maxima and Minima

Methodology:

Algorithm for Finding Maxima and Minima Derivatives can be utilized to locate the extreme values (local maxima and minima) of a function.
Step 1: Find the first derivative $f'(x)$. **Step 2:** Set $f'(x) = 0$ and solve for x to find all stationary points. Let these points be $x = a$ and $x = b \ldots$ **Step 3:** Find the second derivative $f''(x)$. **Step 4:** Substitute each stationary point into the second derivative $f''(x)$ to determine its nature:

- If $f''(a) < 0$ the function has a local **maximum** at $x = a$.
- If $f''(a) > 0$ the function has a local **minimum** at $x = a$.
- If $f''(a) = 0$ the test is inconclusive.

Step 5: Substitute the relevant values of x back into the original function $f(x)$ to calculate the maximum or minimum function values.

Worked Example:

Find maximum and minimum values of $f(x) = x^3 - 3x + 2$ **Step 1: Find the first derivative.**

$$f'(x) = \frac{d}{dx}(x^3 - 3x + 2) = 3x^2 - 3$$

Step 2: Find stationary points by setting $f'(x) = 0$.

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$x^2 = 1 \implies x = 1 \text{ or } x = -1$$

Step 3: Find the second derivative.

$$f''(x) = \frac{d}{dx}(3x^2 - 3) = 6x$$

Step 4: Test each stationary point using the second derivative.
For $x = 1$:

$$f''(1) = 6(1) = 6 > 0$$

Since $f''(1) > 0$ the function achieves a local **minimum** at $x = 1$.
For $x = -1$:

$$f''(-1) = 6(-1) = -6 < 0$$

Since $f''(-1) < 0$ the function achieves a local **maximum** at $x = -1$.
Step 5: Calculate the maximum and minimum values.
Substitute $x = 1$ into the original function for the minimum value:

$$f(1) = (1)^3 - 3(1) + 2 = 1 - 3 + 2 = 0$$

The minimum value is 0.
Substitute $x = -1$ into the original function for the maximum value:

$$f(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$$

The maximum value is 4.

CHAPTER 4

Differentiation and its Applications

4.1 Derivatives using Product and Quotient Rules

Methodology:

How to solve Product and Quotient Rule problems

1. **Identify the form:** Check if the given function y is a product of two functions $u(x) \cdot v(x)$ or a quotient $\frac{u(x)}{v(x)}$.
2. **Product Rule:** If $y = u \cdot v$, apply the formula:
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
3. **Quotient Rule:** If $y = \frac{u}{v}$, apply the formula:
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
4. **Differentiate individual terms:** Use standard derivative formulas for u and v.
5. **Simplify:** Substitute back into the rule and simplify the algebraic expression if possible.

Worked Example:

Quotient Rule: Exponential Function **Problem:** If $y = \frac{e^x+1}{e^x-1}$, then find $\frac{dy}{dx}$. (GTU Summer 2023)
Solution:

1. **Identify u and v:** Let $u = e^x + 1$ and $v = e^x - 1$.
2. **Find derivatives of u and v:**
$$\frac{du}{dx} = \frac{d}{dx}(e^x + 1) = e^x$$
$$\frac{dv}{dx} = \frac{d}{dx}(e^x - 1) = e^x$$
3. **Apply the Quotient Rule:**
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{(e^x - 1)(e^x) - (e^x + 1)(e^x)}{(e^x - 1)^2}$$
4. **Simplify the numerator:** Expand terms in the numerator: $e^{2x} - e^x - (e^{2x} + e^x) = -2e^x$.
5. **Final Answer:**
$$\frac{dy}{dx} = \frac{-2e^x}{(e^x - 1)^2}$$

4.2 Differentiation of Parametric Equations

Methodology:

How to solve Parametric Differentiation problems

1. **Identify the parameters:** You are given $x = f(\theta)$ and $y = g(\theta)$ (or with parameter t).
2. **Differentiate x with respect to the parameter:** Find $\frac{dx}{d\theta}$ using standard differentiation rules.
3. **Differentiate y with respect to the parameter:** Find $\frac{dy}{d\theta}$ similarly.
4. **Apply the Parametric Formula:** Use the rule:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

5. **Simplify:** Substitute the derivatives and simplify the resulting expression using trigonometric identities if necessary.

Worked Example:

Parametric Differentiation: Trigonometric Functions **Problem:** If $x = a\cos\theta$ and $y = b\sin\theta$, then find $\frac{dy}{dx}$. (GTU Summer 2023)
Solution:

1. **Differentiate x w.r.t θ:**

$$x = a\cos\theta \implies \frac{dx}{d\theta} = -a\sin\theta$$

2. **Differentiate y w.r.t θ:**

$$y = b\sin\theta \implies \frac{dy}{d\theta} = b\cos\theta$$

3. **Apply the Formula:**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$$

4. **Simplify:**

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

4.3 Logarithmic Differentiation

Methodology:

How to solve Logarithmic Differentiation problems

1. **Identify the function type:** Used when a function is raised to the power of another function, e.g., $y = u(x)^{v(x)}$, or for complex products/quotients.
2. **Take the natural logarithm (log) on both sides:**

$$\log y = \log(u(x)^{v(x)})$$

3. **Apply log properties:** Move the exponent to the front using $\log(a^b) = b \cdot \log a$.
$$\log y = v(x) \cdot \log(u(x))$$
4. **Differentiate both sides w.r.t x:** Use implicit differentiation on the left: $\frac{1}{y}\frac{dy}{dx}$. Use the Product Rule on the right side.
5. **Solve for $\frac{dy}{dx}$:** Multiply the entire right side by y.
6. **Substitute back for y:** Replace y with the original function to get the final answer strictly in terms of x.

Worked Example:

Function to the Power of a Function **Problem:** Differentiate $y = x^{\cos x}$ with respect to x. (GTU Summer 2023)

Solution:

1. **Take log on both sides:**
$$\log y = \log(x^{\cos x})$$
2. **Apply log properties:**
$$\log y = \cos x \cdot \log x$$
3. **Differentiate both sides w.r.t x:** Apply the product rule on the right side.
$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(\cos x)$$
$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \left(\frac{1}{x}\right) + \log x \cdot (-\sin x)$$
4. **Solve for $\frac{dy}{dx}$:**
$$\frac{dy}{dx} = y\left[\frac{\cos x}{x} - \sin x \cdot \log x\right]$$
5. **Substitute back for y:**
$$\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \cdot \log x\right]$$

4.4 Second-Order Derivatives and Proving Relations

Methodology:

How to prove differential equations using Second-Order Derivatives

1. **Find the first derivative $\left(\frac{dy}{dx}\right)$:** Differentiate the given function y once with respect to x (or t).
2. **Find the second derivative $\left(\frac{d^2y}{dx^2}\right)$:** Differentiate the first derivative again.
3. **Substitute into the target equation:** Plug y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ into the left-hand side (LHS) of the equation you need to prove.
4. **Simplify to match the right-hand side (RHS):** Group terms and use algebraic or trigonometric identities until the LHS simplifies to exactly match the RHS (usually 0).

Worked Example:

Proving a Second-Order Differential Relation **Problem:** If $y = A\cos pt + B\sin pt$, prove that $\frac{d^2y}{dt^2} + p^2y = 0$. (GTU Summer 2023)

Solution:

1. **Find the first derivative $\frac{dy}{dt}$:** Using the chain rule:
$$\frac{dy}{dt} = A(-\sin pt \cdot p) + B(\cos pt \cdot p)$$
$$\frac{dy}{dt} = -Ap\sin pt + Bp\cos pt$$
2. **Find the second derivative $\frac{d^2y}{dt^2}$:** Differentiate $\frac{dy}{dt}$ again w.r.t t:
$$\frac{d^2y}{dt^2} = -Ap(\cos pt \cdot p) + Bp(-\sin pt \cdot p)$$
$$\frac{d^2y}{dt^2} = -Ap^2\cos pt - Bp^2\sin pt$$
3. **Substitute into the LHS of the target equation:** LHS $= \frac{d^2y}{dt^2} + p^2y$
$$\text{LHS} = (-Ap^2\cos pt - Bp^2\sin pt) + p^2(A\cos pt + B\sin pt)$$
4. **Simplify to prove:** Expand the second term:
$$\text{LHS} = -Ap^2\cos pt - Bp^2\sin pt + Ap^2\cos pt + Bp^2\sin pt$$
All terms cancel out.
$$\text{LHS} = 0 = \text{RHS}$$
Hence proved.

4.5 Applications of Derivatives in Kinematics

Methodology:

How to find Velocity and Acceleration from Displacement

1. **Identify the displacement equation:** You are given s (distance or displacement) as a function of time t, i.e., $s = f(t)$.
2. **Find Velocity (v):** Differentiate the displacement equation once with respect to t.
$$v = \frac{ds}{dt}$$
3. **Find Acceleration (a):** Differentiate the velocity equation with respect to t (which is the second derivative of displacement).
$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$
4. **Evaluate at a specific time:** Substitute the given value of t into the velocity and acceleration equations to find their numerical values at that exact moment.

Worked Example:

Calculating Velocity and Acceleration **Problem:** The equation of motion of a particle is $s = t^3 + 2t^2 - 3t + 5$. Find the velocity and acceleration of the particle at $t = 1$ and $t = 2$ seconds. (GTU Summer 2023)
Solution:

1. **Find the general Velocity equation (v):**

$$v = \frac{ds}{dt} = \frac{d}{dt}(t^3 + 2t^2 - 3t + 5)$$

$$v = 3t^2 + 4t - 3$$

2. **Find the general Acceleration equation (a):**

$$a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 4t - 3)$$

$$a = 6t + 4$$

3. **Evaluate at $t = 1$ second:** Velocity: $v(1) = 3(1)^2 + 4(1) - 3 = 3 + 4 - 3 = 4$ units/sec.
   Acceleration: $a(1) = 6(1) + 4 = 10$ units/sec^2.

4. **Evaluate at $t = 2$ seconds:** Velocity: $v(2) = 3(2)^2 + 4(2) - 3 = 12 + 8 - 3 = 17$ units/sec.
   Acceleration: $a(2) = 6(2) + 4 = 12 + 4 = 16$ units/sec^2.

4.6 Maxima and Minima of a Function

Methodology:

How to find Maxima and Minima

1. **Find the first derivative ($f'(x)$):** Differentiate the given function $y = f(x)$.
2. **Find critical points:** Set $f'(x) = 0$ and solve for x. These are the values where a maximum or minimum could occur.
3. **Find the second derivative ($f''(x)$):** Differentiate $f'(x)$ to get $f''(x)$.
4. **Apply the Second Derivative Test:** Substitute each critical point into $f''(x)$:
   - If $f''(x) > 0$, the function has a **minimum** at that point.
   - If $f''(x) < 0$, the function has a **maximum** at that point.
   - If $f''(x) = 0$, the test is inconclusive.
5. **Calculate maximum/minimum values:** Plug the critical x values back into the original function $f(x)$ to find the max/min y values.

Worked Example:

Finding Local Maximum and Minimum **Problem:** Find the maximum and minimum values of the function $f(x) = x^3 - 3x + 2$.
Solution:

1. **Find the first derivative:**

$$f'(x) = \frac{d}{dx}(x^3 - 3x + 2) = 3x^2 - 3$$

2. **Find critical points:** Set $f'(x) = 0$:

$$3x^2 - 3 = 0 \implies 3(x^2 - 1) = 0 \implies x^2 = 1$$

So, $x = 1$ and $x = -1$.

3. **Find the second derivative:**

$$f''(x) = \frac{d}{dx}(3x^2 - 3) = 6x$$

4. **Test each critical point:**

   - At $x = 1$: $f''(1) = 6(1) = 6 > 0$. Thus, $f(x)$ has a **minimum** at $x = 1$.
   - At $x = -1$: $f''(-1) = 6(-1) = -6 < 0$. Thus, $f(x)$ has a **maximum** at $x = -1$.

5. **Find the maximum and minimum values:** Minimum value (at $x = 1$):

$$f(1) = (1)^3 - 3(1) + 2 = 1 - 3 + 2 = 0$$

Maximum value (at $x = -1$):

$$f(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$$

CHAPTER 5

Integration and its Applications

5.1 Basic Rules of Integration

Methodology:

Standard Formulas of Integration Integration is the reverse process of differentiation. The standard formulas are:

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ (where $n \neq -1$)
2. $\int \frac{1}{x} \, dx = \log_e |x| + c$
3. $\int e^x \, dx = e^x + c$
4. $\int a^x \, dx = \frac{a^x}{\log_e a} + c$
5. $\int \sin x \, dx = -\cos x + c$
6. $\int \cos x \, dx = \sin x + c$
7. $\int \sec^2 x \, dx = \tan x + c$
8. $\int \csc^2 x \, dx = -\cot x + c$
9. $\int \sec x \tan x \, dx = \sec x + c$
10. $\int \csc x \cot x \, dx = -\csc x + c$

Worked Example:

Evaluate an algebraic and exponential integral **Problem:** Evaluate the integral $\int \left(3x^2 + \frac{2}{x} - 4e^x\right) dx$
Step 1: Separate the integral into individual terms.

$$\int \left(3x^2 + \frac{2}{x} - 4e^x\right) dx = 3\int x^2 \, dx + 2\int \frac{1}{x} \, dx - 4\int e^x \, dx$$

Step 2: Apply standard integration formulas.

$$= 3\left(\frac{x^3}{3}\right) + 2\log_e |x| - 4e^x + c$$

Step 3: Simplify the expression.

$$= x^3 + 2\log_e |x| - 4e^x + c$$

5.2 Integration by Substitution

Methodology:

Integration by Substitution Method When the integrand is of the form $f(g(x))g'(x)$, use the substitution method:

1. Identify a part of the integrand to substitute. Let $t = g(x)$.
2. Differentiate to find $dt = g'(x)dx$.
3. Replace $g(x)$ with t and $g'(x)dx$ with dt.
4. Integrate with respect to t.
5. Substitute back $t = g(x)$ into the final answer.

Worked Example:

Evaluate integral using algebraic substitution **Problem:** Evaluate $\int \frac{2x}{x^2+1}\,dx$
Step 1: Choose substitution. Let $t = x^2 + 1$.
Step 2: Differentiate. Differentiating with respect to x:

$$\frac{dt}{dx} = 2x \implies dt = 2x\,dx$$

Step 3: Substitute into the integral.

$$\int \frac{1}{x^2+1}(2x\,dx) = \int \frac{1}{t}\,dt$$

Step 4: Integrate with respect to t.

$$= \log_e |t| + c$$

Step 5: Substitute back x.

$$= \log_e |x^2 + 1| + c$$

Worked Example:

Evaluate integral using trigonometric substitution **Problem:** Evaluate $\int e^{\sin x} \cos x\,dx$
Step 1: Choose substitution. Let $t = \sin x$.
Step 2: Differentiate.

$$dt = \cos x\,dx$$

Step 3: Substitute.

$$\int e^t\,dt$$

Step 4: Integrate.

$$= e^t + c$$

Step 5: Substitute back.

$$= e^{\sin x} + c$$

5.3 Integration by Parts

Methodology:

Integration by Parts Formula Used for integrating the product of two functions.

$$\int u \cdot v\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx \right) dx$$

LIATE Rule: Choose the first function (u) in the order of:

1. **L**ogarithmic functions ($\log x$)
2. **I**nverse trigonometric functions ($\sin^{-1} x$)
3. **A**lgebraic functions (x^2 and x)
4. **T**rigonometric functions ($\sin x$ and $\cos x$)
5. **E**xponential functions (e^x)

Worked Example:

Evaluate integral of product of algebraic and trigonometric functions **Problem:** Evaluate $\int x \cos x\, dx$
Step 1: Identify u and v using LIATE. Here x is Algebraic (A) and $\cos x$ is Trigonometric (T). So $u = x$ and $v = \cos x$.
Step 2: Apply the Integration by Parts formula.

$$\int x \cos x\, dx = x \int \cos x\, dx - \int \left(\frac{d}{dx}(x) \int \cos x\, dx \right) dx$$

Step 3: Evaluate integrals and derivatives.

$$= x(\sin x) - \int (1)(\sin x)\, dx$$

Step 4: Integrate the remaining term.

$$= x \sin x - (-\cos x) + c$$

$$= x \sin x + \cos x + c$$

Worked Example:

Evaluate integral using repeated Integration by Parts **Problem:** Evaluate $\int x^2 e^x\, dx$
Step 1: Identify u and v. Algebraic x^2 comes before Exponential e^x. So $u = x^2$ and $v = e^x$.
Step 2: Apply formula.

$$\int x^2 e^x\, dx = x^2 \int e^x\, dx - \int \left(\frac{d}{dx}(x^2) \int e^x\, dx \right) dx$$

$$= x^2 e^x - \int 2x e^x\, dx$$

Step 3: Apply integration by parts again for the new integral. For $\int 2xe^x\, dx$: let $u = 2x$ and $v = e^x$.

$$\int 2xe^x\, dx = 2xe^x - \int 2e^x\, dx = 2xe^x - 2e^x$$

Step 4: Combine the results.

$$\int x^2 e^x\, dx = x^2 e^x - (2xe^x - 2e^x) + c$$

$$= e^x(x^2 - 2x + 2) + c$$

5.4 Definite Integrals

Methodology:

Fundamental Theorem of Calculus If $F(x)$ is the antiderivative of $f(x)$ then:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Properties of Definite Integrals:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
3. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Worked Example:

Evaluate simple definite integral **Problem:** Evaluate $\int_1^2 (x^2 + 2x)\,dx$
Step 1: Find the indefinite integral.

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2$$

Step 2: Apply limits.

$$\int_1^2 (x^2 + 2x)\,dx = \left[\frac{x^3}{3} + x^2\right]_1^2$$

Step 3: Substitute upper limit and lower limit.

$$= \left(\frac{2^3}{3} + 2^2\right) - \left(\frac{1^3}{3} + 1^2\right)$$

$$= \left(\frac{8}{3} + 4\right) - \left(\frac{1}{3} + 1\right)$$

Step 4: Simplify.

$$= \left(\frac{20}{3}\right) - \left(\frac{4}{3}\right) = \frac{16}{3}$$

Worked Example:

Evaluate definite integral using property **Problem:** Evaluate $\int_0^{\pi/2} \sin^2 x\,dx$
Let $I = \int_0^{\pi/2} \sin^2 x\,dx \ldots$(Equation 1)
Step 1: Apply property involving difference from upper limit. Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$:

$$I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_0^{\pi/2} \cos^2 x\,dx$$

$\ldots$(Equation 2)
Step 2: Add Equation 1 and Equation 2.

$$I + I = \int_0^{\pi/2} \sin^2 x\,dx + \int_0^{\pi/2} \cos^2 x\,dx$$

$$2I = \int_0^{\pi/2} (\sin^2 x + \cos^2 x)\, dx$$

Step 3: Simplify and integrate. Since $\sin^2 x + \cos^2 x = 1$:

$$2I = \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

Step 4: Solve for I.

$$I = \frac{\pi}{4}$$

5.5 Applications of Definite Integrals

5.5.1 Area Under a Curve

Methodology:

Finding Area Under a Curve The area A bounded by the curve $y = f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$ is given by:

$$A = \int_a^b |y|\, dx = \int_a^b |f(x)|\, dx$$

For regions above the x-axis, the integral evaluates directly to the physical area.

Worked Example:

Find area bounded by parabola and x-axis **Problem:** Find the area bounded by the curve $y = x^2$ and the x-axis between $x = 1$ and $x = 3$.

Step 1: Set up the definite integral.

$$A = \int_a^b y\, dx = \int_1^3 x^2\, dx$$

Step 2: Integrate the function.

$$\int x^2\, dx = \frac{x^3}{3}$$

Step 3: Evaluate the definite integral.

$$A = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3}$$

$$= \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

The area is $\frac{26}{3}$ square units.

5.5.2 Volume of Solid of Revolution

Methodology:

Finding Volume of Solid of Revolution The volume V of a solid generated by revolving the region bounded by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$ about the x-axis is:

$$V = \pi \int_a^b y^2\, dx = \pi \int_a^b [f(x)]^2\, dx$$

Worked Example:

Find volume of solid generated by revolving curve about x-axis **Problem:** Find the volume of the solid generated by revolving the curve $y = \sqrt{x}$ from $x = 0$ to $x = 4$ about the x-axis.
Step 1: Set up the volume integral.

$$V = \pi \int_a^b y^2 \, dx$$

Here $y = \sqrt{x}$ so $y^2 = x$. The limits are $a = 0$ and $b = 4$.

$$V = \pi \int_0^4 x \, dx$$

Step 2: Integrate.

$$\int x \, dx = \frac{x^2}{2}$$

Step 3: Apply the limits.

$$V = \pi \left[\frac{x^2}{2} \right]_0^4 = \pi \left(\frac{4^2}{2} - \frac{0^2}{2} \right)$$
$$= \pi \left(\frac{16}{2} - 0 \right) = 8\pi$$

The volume is 8π cubic units.

5.5.3 Mean and RMS Values

Methodology:

Mean and RMS Value of a Function For a function $f(t)$ defined for $a \le t \le b$:

1. **Mean (Average) Value:**
$$f_{\text{mean}} = \frac{1}{b-a} \int_a^b f(t) \, dt$$
2. **Root Mean Square (RMS) Value:**
$$f_{\text{rms}} = \sqrt{\frac{1}{b-a} \int_a^b [f(t)]^2 \, dt}$$

Worked Example:

Find mean and RMS value of sine function **Problem:** Find the mean and RMS value of $f(t) = \sin t$ over the interval $[0 \ldots \pi]$.
Step 1: Find the Mean Value.

$$f_{\text{mean}} = \frac{1}{\pi - 0} \int_0^\pi \sin t \, dt$$
$$= \frac{1}{\pi} [-\cos t]_0^\pi = \frac{1}{\pi} (-\cos \pi - (-\cos 0))$$
$$= \frac{1}{\pi} (-(-1) + 1) = \frac{2}{\pi}$$

Step 2: Find the integral of the square of the function.

$$\int_0^\pi \sin^2 t \, dt = \int_0^\pi \frac{1 - \cos 2t}{2} \, dt$$
$$= \frac{1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^\pi$$
$$= \frac{1}{2} \left(\left(\pi - \frac{\sin 2\pi}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right)$$

$$= \frac{1}{2}(\pi - 0 - 0 + 0) = \frac{\pi}{2}$$

Step 3: Calculate the RMS Value.

$$f_{\mathrm{rms}} = \sqrt{\frac{1}{\pi - 0}\left(\frac{\pi}{2}\right)} = \sqrt{\frac{1}{\pi} \cdot \frac{\pi}{2}} = \frac{1}{\sqrt{2}}$$

CHAPTER 6

Integration and its Applications

6.1 Evaluating Basic Indefinite Integrals

Methodology:

How to evaluate basic integrals using standard formulas 1. Identify the form of the function in the integrand. 2. Use standard direct formulas:

- Power Rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- Constant: $\int 1\,dx = x + C$
- Exponential: $\int e^x\,dx = e^x + C$
- Trigonometric: $\int \sin x\,dx = -\cos x + C$, $\int \cos x\,dx = \sin x + C$
- Inverse Trigonometric Form: $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

3. If the integrand is a sum or difference of terms, apply the integral to each term separately. 4. Add the constant of integration C at the end.

Worked Example:

Direct Formula Application Evaluate $\int \frac{1}{1+x^2}\,dx$.
Solution:

1. Identify that the integrand exactly matches the standard formula for the derivative of $\tan^{-1} x$.
2. Apply the direct formula:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

Worked Example:

Basic Trigonometric Integral Evaluate $\int \sin x\,dx$.
Solution:

1. Identify the standard trigonometric function.
2. Apply the integral formula for $\sin x$:

$$\int \sin x\,dx = -\cos x + C$$

6.2 Integration by Substitution

Methodology:

How to evaluate integrals using the substitution method 1. Identify a part of the integrand (often the "inner" function or denominator) to act as a new variable, say $u = g(x)$. A good choice for u is a function whose derivative $g'(x)$ is also present in the integrand. 2. Differentiate u with respect to x to find $du = g'(x)dx$. 3. Substitute u and du into the original integral to completely eliminate x and dx. 4. Evaluate the new, simpler integral with respect to u. 5. Replace u with the original expression in x to write the final answer.

Worked Example:

Substitution with a Square Root Evaluate $\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$.
Solution:

1. Choose $u = \sqrt{x}$ since its derivative is related to the denominator.
2. Differentiate u:
$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \implies du = \frac{1}{2\sqrt{x}}\,dx$$
3. Notice that $\frac{1}{2\sqrt{x}}\,dx$ is exactly present in the given integral. Substitute u and du:
$$\int \cos(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}}\,dx\right) = \int \cos u\,du$$
4. Evaluate the integral with respect to u:
$$\int \cos u\,du = \sin u + C$$
5. Replace u with $\sqrt{x}$ to get the final answer:
$$= \sin(\sqrt{x}) + C$$

6.3 Integration by Parts

Methodology:

How to evaluate the integral of a product of two functions 1. Check if the integrand is a product of two distinct types of functions (e.g., algebraic and logarithmic). 2. Assign the functions to u (which will be differentiated) and dv (which will be integrated). 3. Use the **LIATE** rule to choose u (pick the function that appears first in the list):

- **L**ogarithmic (e.g., $\log x$)
- **I**nverse trigonometric
- **A**lgebraic (e.g., x, x^2)
- **T**rigonometric (e.g., $\sin x, \cos x$)
- **E**xponential (e.g., e^x)

4. Find du by differentiating u, and find v by integrating dv. 5. Apply the Integration by Parts formula:

$$\int u\,dv = uv - \int v\,du$$

6. Evaluate the remaining integral $\int v\,du$ and add the constant C.

Worked Example:

Product of Algebraic and Logarithmic Functions Evaluate $\int x \cdot \log x\,dx$.
Solution:

1. Identify the two functions: x (Algebraic) and $\log x$ (Logarithmic).
2. According to LIATE, Logarithmic comes before Algebraic. So, choose $u = \log x$ and $dv = x\,dx$.
3. Differentiate u and integrate dv:

$$u = \log x \implies du = \frac{1}{x}\,dx$$

$$dv = x\,dx \implies v = \int x\,dx = \frac{x^2}{2}$$

4. Apply the Integration by Parts formula $\int u\,dv = uv - \int v\,du$:

$$\int x \log x\,dx = (\log x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)\,dx$$

5. Simplify and solve the remaining integral:

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + C$$

6.4 Evaluating Definite Integrals

Methodology:

How to evaluate standard definite integrals 1. Find the indefinite integral of the function without adding the constant C. Let this be $F(x)$. 2. Use square brackets to write the upper and lower limits: $[F(x)]_a^b$. 3. Substitute the upper limit b into $F(x)$. 4. Substitute the lower limit a into $F(x)$. 5. Subtract the lower limit result from the upper limit result: $F(b) - F(a)$.

Worked Example:

Evaluating a Polynomial Definite Integral Evaluate $\int_{-1}^{1} x^3\,dx$.
Solution:

1. Find the indefinite integral of x^3:
$$\int x^3\,dx = \frac{x^4}{4}$$
2. Apply the upper and lower limits:
$$\int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1}$$
3. Substitute the upper limit (1) and lower limit (-1) and subtract:
$$= \left(\frac{(1)^4}{4}\right) - \left(\frac{(-1)^4}{4}\right)$$
$$= \frac{1}{4} - \frac{1}{4}$$
$$= 0$$

Worked Example:

Evaluating an Inverse Trigonometric Definite Integral Evaluate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$.
Solution:

1. The integral of $\frac{1}{1+x^2}$ is $\tan^{-1} x$.
2. Apply the limits:
$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_{-1}^{1}$$
3. Substitute the upper and lower limits:
$$= \tan^{-1}(1) - \tan^{-1}(-1)$$
4. Since $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$:
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Methodology:

How to evaluate definite integrals using the symmetric property This method is extremely useful for integrals containing trigonometric ratios in fractions where the denominator remains the same after applying the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. 1. Let the given integral be I. Label this as Equation (1). 2. Apply the property: Replace every x in the integrand with $(a-x)$, where a is the upper limit. 3. Simplify the new integrand using trigonometric identities (e.g., $\tan(\frac{\pi}{2}-x) = \cot x$). Label this new expression for I as Equation (2). 4. Add Equation (1) and Equation (2). This gives $2I$ on the left side, and the integrand on the right side usually simplifies drastically (often to 1). 5. Integrate the simplified right side and divide by 2 to find I.

Worked Example:

Definite Integral using Properties Evaluate $\int_0^{\frac{\pi}{2}} \frac{\tan x}{\tan x+\cot x}\,dx$.
Solution:

1. Let the given integral be I:
$$I = \int_0^{\frac{\pi}{2}} \frac{\tan x}{\tan x + \cot x}\,dx \quad \dots(1)$$
2. Apply the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Replace x with $\left(\frac{\pi}{2}-x\right)$:
$$I = \int_0^{\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{\tan\left(\frac{\pi}{2}-x\right) + \cot\left(\frac{\pi}{2}-x\right)}\,dx$$
3. Simplify using the identities $\tan(\frac{\pi}{2}-x) = \cot x$ and $\cot(\frac{\pi}{2}-x) = \tan x$:
$$I = \int_0^{\frac{\pi}{2}} \frac{\cot x}{\cot x + \tan x}\,dx \quad \dots(2)$$
4. Add Equation (1) and Equation (2):
$$I + I = \int_0^{\frac{\pi}{2}} \frac{\tan x}{\tan x + \cot x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cot x}{\tan x + \cot x}\,dx$$
$$2I = \int_0^{\frac{\pi}{2}} \frac{\tan x + \cot x}{\tan x + \cot x}\,dx$$
$$2I = \int_0^{\frac{\pi}{2}} 1\,dx$$
5. Integrate and solve for I:
$$2I = [x]_0^{\frac{\pi}{2}}$$
$$2I = \frac{\pi}{2} - 0$$
$$I = \frac{\pi}{4}$$

6.5 Applications of Definite Integrals: Area Bounded by Curves

Methodology:

How to find the area bounded by a curve and an axis 1. Identify the equation of the curve $y = f(x)$. 2. Identify the boundary lines (limits), e.g., $x = a$ and $x = b$. If one limit is not given, check if the curve starts from the origin (like the intersection with the Y-axis where $x = 0$). 3. The area A bounded by the curve and the X-axis between $x = a$ and $x = b$ is given by the formula:

$$A = \int_a^b y\, dx$$

4. Substitute y with $f(x)$ and evaluate the definite integral.

Worked Example:

Finding Area Bounded by a Line and X-axis Find the area bounded by the line $y = x$, $x = 5$ and the X-axis.
Solution:

1. The curve (line) equation is $y = x$.
2. The boundary limit on the right is $x = 5$. Since it's bounded by the X-axis and $x = 5$, the region starts from the origin $x = 0$. So, the limits are from $x = 0$ to $x = 5$.
3. Set up the area integral formula:
$$A = \int_0^5 y\, dx$$
4. Substitute $y = x$ and evaluate:
$$A = \int_0^5 x\, dx$$
$$A = \left[\frac{x^2}{2}\right]_0^5$$
$$A = \frac{(5)^2}{2} - \frac{(0)^2}{2}$$
$$A = \frac{25}{2} - 0 = \frac{25}{2} \text{ square units}$$

CHAPTER 7

Differential Equations

7.1 Order and Degree of a Differential Equation

Methodology:

Finding Order and Degree 1. **Order:** The order of a differential equation is the highest derivative present in the equation. 2. **Degree:** The degree is the power of the highest order derivative, provided the equation is written as a polynomial in derivatives (i.e. free from fractional powers and radicals). 3. If derivatives are trapped inside transcendental functions (like $\sin(y')$, $e^{y'}$, or $\log(y')$), the degree is not defined.

Worked Example:

Find the order and degree of second order derivative equations **Problem:** Find the order and degree of $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = 4y$.

Step 1: Identify the highest order derivative. The highest derivative is $\frac{d^2y}{dx^2}$, which has an order of 2. Therefore, Order $= 2$.

Step 2: Identify the power of the highest order derivative. The equation is free from radicals. The power of the highest derivative $\frac{d^2y}{dx^2}$ is 1. Therefore, Degree $= 1$.

Worked Example:

Find the order and degree involving radicals **Problem:** Find the order and degree of $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$.

Step 1: Remove radicals. Square both sides of the equation to remove the square root and make it a polynomial in derivatives:

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$$

Step 2: Identify order and degree. The highest order derivative is $\frac{d^2y}{dx^2}$. Order $= 2$. The power of this highest order derivative is 2. Degree $= 2$.

7.2 Formation of Differential Equations

Methodology:

Forming a Differential Equation 1. Given an equation with variables x, y and arbitrary constants (A, B, etc.). 2. Count the number of arbitrary constants. Let this number be n. 3. Differentiate the given equation successively n times to obtain n new equations involving derivatives. 4. Eliminate the n arbitrary constants using the original equation and the differentiated equations. 5. The resulting equation free of arbitrary constants is the required differential equation. Its order equals the number of arbitrary constants eliminated.

Worked Example:

Form the differential equation for an exponential family **Problem:** Form the differential equation representing the family of curves $y = Ae^{2x} + Be^{-2x}$.
Step 1: Count arbitrary constants. There are two arbitrary constants, A and B. Therefore, we will differentiate twice.
Step 2: Differentiate successively. First derivative:

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$$

Second derivative:

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$$

Step 3: Eliminate constants. Factor out 4 from the second derivative:

$$\frac{d^2y}{dx^2} = 4(Ae^{2x} + Be^{-2x})$$

Notice that the term in the parenthesis is the original equation y. Substituting y:

$$\frac{d^2y}{dx^2} = 4y$$

This is the required differential equation.

7.3 Variable Separable Method

Methodology:

Solving by Variable Separable Method 1. Given a first-order differential equation, use algebraic manipulation to bring all terms containing y and dy to one side, and all terms containing x and dx to the other side. 2. The equation should take the form: $g(y)dy = f(x)dx$. 3. Integrate both sides: $\int g(y)dy = \int f(x)dx + C$. 4. Simplify the resulting equation to find the general solution.

Worked Example:

Solve a basic variable separable equation **Problem:** Solve the differential equation $\frac{dy}{dx} = x^2y$.
Step 1: Separate the variables. Divide both sides by y and multiply by dx:

$$\frac{dy}{y} = x^2\,dx$$

Step 2: Integrate both sides.

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

Step 3: Simplify the solution. Express y explicitly by taking the exponential of both sides:

$$y = e^{\frac{x^3}{3}+C} = e^C \cdot e^{\frac{x^3}{3}}$$

Let $A = e^C$, giving the general solution:

$$y = Ae^{\frac{x^3}{3}}$$

Worked Example:

Solve an inverse trigonometric separable equation **Problem:** Solve $(1+x^2)dy - (1+y^2)dx = 0$.
Step 1: Separate the variables. Rearrange the terms:

$$(1+x^2)dy = (1+y^2)dx$$

Divide by $(1+x^2)(1+y^2)$:

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Step 2: Integrate both sides.

$$\int \frac{1}{1+y^2}dy = \int \frac{1}{1+x^2}dx$$

$$\tan^{-1}(y) = \tan^{-1}(x) + C$$

This is the general solution.

7.4 Homogeneous Differential Equations

Methodology:

Solving Homogeneous Equations 1. Check if the equation can be written as $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous functions of the same degree. 2. Substitute $y = vx$. Differentiate with respect to x to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$. 3. Substitute y and $\frac{dy}{dx}$ into the original equation. The equation will reduce to a variable separable form in x and v. 4. Separate the variables x and v, and integrate both sides. 5. Replace v with $\frac{y}{x}$ in the final integrated expression to get the general solution.

Worked Example:

Solve a rational homogeneous equation **Problem:** Solve the differential equation $\frac{dy}{dx} = \frac{y^2+xy}{x^2}$.
Step 1: Verify homogeneity and substitute. The numerator and denominator are both homogeneous polynomials of degree 2. Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
Step 2: Substitute into the differential equation.

$$v + x\frac{dv}{dx} = \frac{(vx)^2 + x(vx)}{x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2(v^2+v)}{x^2} = v^2 + v$$

Step 3: Separate variables. Cancel v from both sides:

$$x\frac{dv}{dx} = v^2$$

$$\frac{dv}{v^2} = \frac{dx}{x}$$

Step 4: Integrate both sides.

$$\int v^{-2}dv = \int \frac{1}{x}dx$$

$$-\frac{1}{v} = \ln|x| + C$$

Step 5: Substitute $v = y/x$ back.

$$-\frac{x}{y} = \ln|x| + C$$

$$y = \frac{-x}{\ln|x| + C}$$

7.5 Linear Differential Equations

Methodology:

Solving Linear Differential Equations 1. Write the differential equation in the standard linear form: $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x alone or constants. 2. Calculate the Integrating Factor (IF) using the formula: $\text{IF} = e^{\int P(x)dx}$. 3. Write the general solution using the formula: $y \cdot (\text{IF}) = \int Q(x) \cdot (\text{IF})dx + C$. 4. Evaluate the integral on the right-hand side to find the final solution.

Worked Example:

Solve a linear DE with exponential integrating factor **Problem:** Solve the differential equation $\frac{dy}{dx} + 2xy = x$.
Step 1: Identify standard form and determine P and Q. The equation is already in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$. Here, $P(x) = 2x$ and $Q(x) = x$.
Step 2: Find the Integrating Factor (IF).

$$\text{IF} = e^{\int P(x)dx} = e^{\int 2xdx} = e^{x^2}$$

Step 3: Apply the general solution formula.

$$y \cdot (\text{IF}) = \int Q(x) \cdot (\text{IF})dx + C$$

$$y \cdot e^{x^2} = \int x \cdot e^{x^2} dx + C$$

Step 4: Evaluate the integral. Let $t = x^2$, then $dt = 2xdx$, or $xdx = \frac{dt}{2}$.

$$\int xe^{x^2} dx = \int e^t \frac{dt}{2} = \frac{1}{2}e^t = \frac{1}{2}e^{x^2}$$

Substitute back into the general solution:

$$y \cdot e^{x^2} = \frac{1}{2}e^{x^2} + C$$

Divide by e^{x^2}:

$$y = \frac{1}{2} + Ce^{-x^2}$$

Worked Example:

Solve a linear DE requiring division by x **Problem:** Solve $x\frac{dy}{dx} + 2y = x^2$.
Step 1: Convert to standard form. Divide the entire equation by x:

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

Here, $P(x) = \frac{2}{x}$ and $Q(x) = x$.
Step 2: Find the Integrating Factor (IF).

$$\text{IF} = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$$

Step 3: Apply the general solution formula.

$$y \cdot (\text{IF}) = \int Q(x) \cdot (\text{IF})dx + C$$

$$y \cdot x^2 = \int (x)(x^2)dx + C$$

Step 4: Evaluate the integral.

$$yx^2 = \int x^3 dx + C$$

$$yx^2 = \frac{x^4}{4} + C$$

Divide by x^2:

$$y = \frac{x^2}{4} + \frac{C}{x^2}$$

CHAPTER 8

Differential Equations

8.1 Order and Degree of a Differential Equation

Methodology:

How to Find the Order and Degree of a Differential Equation To determine the order and degree of a given differential equation, follow this step-by-step procedure:

1. **Identify the Order:** Locate the highest derivative present in the equation (e.g., $\frac{dy}{dx}$ is 1st order, $\frac{d^2y}{dx^2}$ is 2nd order, $\frac{d^3y}{dx^3}$ is 3rd order). The order of this highest derivative is the **order** of the differential equation.

2. **Remove Radicals and Fractions:** Ensure the equation is free from fractional powers (like 1/2, 3/2) on any of the derivatives. If there are fractional powers, square or cube both sides of the equation to clear them.

3. **Identify the Degree:** Once the equation is expressed as a polynomial in derivatives, find the power (exponent) of the highest order derivative identified in Step 1. This power is the **degree** of the differential equation.

4. *Note on Undefined Degree:* If the differential equation contains derivatives inside transcendental functions (e.g., $\sin(dy/dx)$, $e^{dy/dx}$, or $\ln(dy/dx)$), the equation cannot be written as a polynomial in its derivatives, and the degree is **not defined**.

Worked Example:

Finding Order and Degree (Basic Form) Find the order and degree of the following differential equation:

$$\left(\frac{d^3y}{dx^3}\right)^2 + 4\left(\frac{dy}{dx}\right)^4 - 6y = 0$$

Solution:

1. **Identify highest derivative:** The derivatives present in the equation are $\frac{d^3y}{dx^3}$ (3rd order) and $\frac{dy}{dx}$ (1st order). The highest order derivative is $\frac{d^3y}{dx^3}$. Therefore, the **order is 3**.

2. **Check for radicals:** The equation already has integer powers for all derivatives. No manipulation is needed.

3. **Find the power of the highest derivative:** The term with the highest derivative is $\left(\frac{d^3y}{dx^3}\right)^2$. Its exponent is 2. Therefore, the **degree is 2**.

Final Answer: Order = 3, Degree = 2.

Worked Example:

Finding Order and Degree with Fractional Powers Find the order and degree of the following differential equation:

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

Solution:

1. **Remove radicals:** The right side of the equation has a fractional power of $\frac{3}{2}$. To remove this fraction, we square both sides of the equation:

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

2. **Identify highest derivative:** Looking at the simplified equation, the highest derivative is $\frac{d^2y}{dx^2}$. Thus, the **order is 2**.

3. **Find the power of the highest derivative:** In the simplified equation, the highest order derivative term is $\left(\frac{d^2y}{dx^2}\right)^2$. The exponent is 2. Thus, the **degree is 2**.

Final Answer: Order = 2, Degree = 2.

8.2 Solution of First Order DE by Variable Separable Method

Methodology:

How to Solve by the Variable Separable Method Use this method when you can algebraically completely separate all terms containing x (including dx) to one side of the equation and all terms containing y (including dy) to the other side.

1. **Rearrange to separate variables:** Manipulate the given equation into the form $f(x)dx = g(y)dy$. Use cross-multiplication, factoring, or division to isolate x-terms with dx and y-terms with dy.

2. **Integrate both sides:** Apply the integration symbol to both sides of the separated equation:

$$\int g(y)\,dy = \int f(x)\,dx$$

3. **Evaluate the integrals:** Solve the integration on both sides using standard integral formulas.

4. **Add the constant of integration ($+C$):** Add a single arbitrary constant C to one side (conventionally to the side with the independent variable x).

5. **Simplify the equation:** Rearrange the final expression to explicitly solve for y if possible, or leave it in simplified implicit form. If an initial condition is given (e.g., $y(0) = 1$), substitute these values to find the exact value of C.

Worked Example:

Solving a Basic Variable Separable Equation Solve the differential equation: $\frac{dy}{dx} = \frac{x^2}{y}$.
Solution:

1. **Rearrange to separate variables:** Cross-multiply to pair y with dy and x^2 with dx:

$$y\,dy = x^2\,dx$$

2. **Integrate both sides:**

$$\int y\,dy = \int x^2\,dx$$

3. **Evaluate the integrals:** Using the standard power rule $\int x^n dx = \frac{x^{n+1}}{n+1}$:

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

4. **Simplify the equation:** Multiply the entire equation by 6 to clear the fractions:

$$3y^2 = 2x^3 + 6C$$

Let $C_1 = 6C$ as a new arbitrary constant.

$$3y^2 - 2x^3 = C_1$$

Final Answer: The general solution is $3y^2 - 2x^3 = C_1$.

Worked Example:

Solving with Exponential Terms Solve the differential equation: $\frac{dy}{dx} = e^{x-y}$.
Solution:

1. **Rearrange to separate variables:** First, use the laws of exponents to rewrite e^{x-y} as $\frac{e^x}{e^y}$.

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$

Cross-multiply to separate the variables:

$$e^y\,dy = e^x\,dx$$

2. **Integrate both sides:**

$$\int e^y\,dy = \int e^x\,dx$$

3. **Evaluate the integrals:** Using the standard integral $\int e^u du = e^u$:

$$e^y = e^x + C$$

4. **Simplify the equation:** You can leave the answer in this form or optionally take the natural logarithm of both sides to solve explicitly for y:

$$y = \ln(e^x + C)$$

Final Answer: The general solution is $e^y = e^x + C$.

8.3 Solution of Linear Differential Equation

Methodology:

How to Solve a Linear Differential Equation Use this method for equations that are (or can be written) in the standard linear form: $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x alone or constants.

1. **Convert to standard form:** Rearrange the given differential equation to match $\frac{dy}{dx} + P(x)y = Q(x)$. Ensure the coefficient in front of $\frac{dy}{dx}$ is strictly 1. If it is not 1, divide the entire equation by that

coefficient.

2. **Identify $P(x)$ and $Q(x)$:** Clearly extract the function multiplying y as $P(x)$ and the right-hand side function as $Q(x)$.

3. **Calculate the Integrating Factor (I.F.):** Compute the Integrating Factor using the formula:

$$\text{I.F.} = e^{\int P(x)\,dx}$$

Helpful tip: Recall the logarithmic identity $e^{\ln(f(x))} = f(x)$ to simplify the I.F. calculation.

4. **Apply the general solution formula:** Write down the solution template:

$$y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})\,dx + C$$

5. **Evaluate the final integral:** Substitute $Q(x)$ and the computed I.F. into the formula, perform the integration on the right side, and solve explicitly for y.

Worked Example:

Solving a Standard Linear Differential Equation Solve the linear differential equation: $\frac{dy}{dx} + 3y = e^{2x}$.
Solution:

1. **Convert to standard form:** The equation is already in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.

2. **Identify $P(x)$ and $Q(x)$:**

$$P(x) = 3$$
$$Q(x) = e^{2x}$$

3. **Calculate the Integrating Factor (I.F.):**

$$\text{I.F.} = e^{\int P(x)\,dx} = e^{\int 3\,dx} = e^{3x}$$

4. **Apply the general solution formula:**

$$y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})\,dx + C$$

5. **Evaluate the final integral:** Substitute $\text{I.F.} = e^{3x}$ and $Q(x) = e^{2x}$:

$$y \cdot e^{3x} = \int e^{2x} \cdot e^{3x}\,dx + C$$

$$y \cdot e^{3x} = \int e^{5x}\,dx + C$$

$$y \cdot e^{3x} = \frac{e^{5x}}{5} + C$$

To find explicit y, divide the entire equation by e^{3x}:

$$y = \frac{e^{5x}}{5e^{3x}} + \frac{C}{e^{3x}}$$

$$y = \frac{e^{2x}}{5} + Ce^{-3x}$$

Final Answer: $y = \frac{e^{2x}}{5} + Ce^{-3x}$

Worked Example:

Solving a Linear Equation Requiring Rearrangement Solve the differential equation: $x\frac{dy}{dx} + y = x^3$ (where $x > 0$).

Solution:

1. **Convert to standard form:** The coefficient of $\frac{dy}{dx}$ is x. Divide the entire equation by x to make the coefficient 1:
$$\frac{dy}{dx} + \frac{1}{x}y = \frac{x^3}{x}$$
$$\frac{dy}{dx} + \frac{1}{x}y = x^2$$
2. **Identify $P(x)$ and $Q(x)$:**
$$P(x) = \frac{1}{x}$$
$$Q(x) = x^2$$
3. **Calculate the Integrating Factor (I.F.):**
$$\text{I.F.} = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln(x)}$$
Using the property $e^{\ln(x)} = x$, we get:
$$\text{I.F.} = x$$
4. **Apply the general solution formula:**
$$y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})\,dx + C$$
5. **Evaluate the final integral:** Substitute I.F. $= x$ and $Q(x) = x^2$:
$$y \cdot x = \int (x^2) \cdot (x)\,dx + C$$
$$yx = \int x^3\,dx + C$$
$$yx = \frac{x^4}{4} + C$$
Divide by x to isolate y:
$$y = \frac{x^3}{4} + \frac{C}{x}$$

Final Answer: $y = \frac{x^3}{4} + \frac{C}{x}$

CHAPTER 9

Complex Numbers

9.1 Introduction and Basic Arithmetic

A complex number is defined as $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$. The real part is denoted by $\text{Re}(z) = x$ and the imaginary part is denoted by $\text{Im}(z) = y$.

Methodology:

Arithmetic of Complex Numbers Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

1. **Addition:** $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
2. **Subtraction:** $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
3. **Multiplication:** $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
4. **Conjugate:** The conjugate of $z = x + iy$ is $\bar{z} = x - iy$.
5. **Division:** To divide z_1 by z_2, multiply the numerator and denominator by the conjugate of the denominator:
$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2}$$

Worked Example:

Simplify the expression and express it in the form x + iy Evaluate $(3 + 2i)(4 - 5i)$.
Step 1: Apply distributive property.

$$(3 + 2i)(4 - 5i) = 3(4 - 5i) + 2i(4 - 5i)$$

Step 2: Expand the terms.
$$= 12 - 15i + 8i - 10i^2$$

Step 3: Substitute $i^2 = -1$ and simplify.

$$= 12 - 7i - 10(-1)$$

$$= 12 - 7i + 10$$
$$= 22 - 7i$$

Final Answer: $22 - 7i$

Worked Example:

Divide complex numbers Express $\frac{3+4i}{2-i}$ in the form $x + iy$.
Step 1: Identify the conjugate of the denominator. The denominator is $2 - i$. Its conjugate is $2 + i$.

Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{3+4i}{2-i} \cdot \frac{2+i}{2+i}$$

Step 3: Multiply the numerators and denominators. Numerator: $(3+4i)(2+i) = 6+3i+8i+4i^2 = 6+11i-4 = 2+11i$ Denominator: $(2-i)(2+i) = 2^2-(i)^2 = 4-(-1) = 5$
Step 4: Write in the final form.

$$\frac{2+11i}{5} = \frac{2}{5} + i\frac{11}{5}$$

Final Answer: $\frac{2}{5} + \frac{11}{5}i$

9.2 Modulus and Argument

Methodology:

Modulus and Argument of a Complex Number For a complex number $z = x + iy$:

1. **Modulus:** $|z| = r = \sqrt{x^2+y^2}$. The modulus represents the distance of the point (x, y) from the origin.
2. **Argument (Amplitude):** Let θ be the principal argument of z, denoted by $\text{Arg}(z)$, where $-\pi < \theta \leq \pi$.
   - Find the acute angle $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$.
   - Determine the quadrant in which (x, y) lies:
     - **Quadrant I ($x > 0$ and $y > 0$):** $\theta = \alpha$
     - **Quadrant II ($x < 0$ and $y > 0$):** $\theta = \pi - \alpha$
     - **Quadrant III ($x < 0$ and $y < 0$):** $\theta = -\pi + \alpha$ (or $\alpha - \pi$)
     - **Quadrant IV ($x > 0$ and $y < 0$):** $\theta = -\alpha$

Worked Example:

Find Modulus and Argument of a Second Quadrant Complex Number Find the Modulus and Argument of $z = -1 + \sqrt{3}i$.
Step 1: Identify x and y. $x = -1$, $y = \sqrt{3}$.
Step 2: Calculate Modulus.

$$|z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

Step 3: Find acute angle α.

$$\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

Step 4: Determine quadrant and find θ. Since $x < 0$ and $y > 0$, the point lies in Quadrant II.

$$\theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Final Answer: Modulus $r = 2$, Argument $\theta = \frac{2\pi}{3}$.

Worked Example:

Find Modulus and Argument of a Third Quadrant Complex Number Find the Modulus and Argument of $z = -1 - i$.

Step 1: Identify x and y. $x = -1$, $y = -1$.
Step 2: Calculate Modulus.

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

Step 3: Find acute angle α.

$$\alpha = \tan^{-1}\left|\frac{-1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$$

Step 4: Determine quadrant and find θ. Since $x < 0$ and $y < 0$, the point lies in Quadrant III.

$$\theta = -\pi + \alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

Final Answer: Modulus $r = \sqrt{2}$, Argument $\theta = -\frac{3\pi}{4}$.

9.3 Polar and Exponential Forms

Methodology:

Polar and Exponential Forms Once the modulus r and principal argument θ of a complex number $z = x + iy$ are known, it can be expressed in different forms:

1. **Polar Form:** $z = r(\cos\theta + i\sin\theta)$
2. **Exponential Form:** $z = re^{i\theta}$

(Note: By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$)

Worked Example:

Convert to Polar and Exponential Form Express $z = \sqrt{3} + i$ in polar and exponential forms.
Step 1: Find Modulus r. Here, $x = \sqrt{3}$ and $y = 1$.

$$r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = \sqrt{4} = 2$$

Step 2: Find Argument θ.

$$\alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$$

Since $x > 0$ and $y > 0$, the point is in Quadrant I, so $\theta = \alpha = \frac{\pi}{6}$.
Step 3: Write Polar Form.

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Step 4: Write Exponential Form.

$$z = re^{i\theta} = 2e^{i\frac{\pi}{6}}$$

Final Answer: Polar form is $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, Exponential form is $2e^{i\frac{\pi}{6}}$.

9.4 Square Root of a Complex Number

Methodology:

Square Root of a Complex Number To find the square root of a complex number $z = x + iy$, set:

$$\sqrt{x + iy} = a + ib$$

where a and b are real numbers.

1. Square both sides: $x + iy = (a^2 - b^2) + i(2ab)$
2. Equate real and imaginary parts:
$$a^2 - b^2 = x$$
$$2ab = y$$
3. Use the algebraic identity: $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$ to find $a^2 + b^2 = \sqrt{x^2 + y^2} = |z|$.
4. Solve the system of equations for a^2 and b^2:
$$a^2 - b^2 = x$$
$$a^2 + b^2 = |z|$$
Adding gives $2a^2 = |z| + x$, subtracting gives $2b^2 = |z| - x$.
5. Find a and b. The sign of ab depends on the sign of y. If $y > 0$, a and b have the same sign. If $y < 0$, a and b have opposite signs.

Worked Example:

Find the square root of a complex number Find $\sqrt{3 - 4i}$.
Step 1: Set up the equation. Let $\sqrt{3 - 4i} = a + ib$. Squaring both sides gives:

$$3 - 4i = (a^2 - b^2) + i(2ab)$$

Step 2: Equate parts. Real part: $a^2 - b^2 = 3$
Imaginary part: $2ab = -4$
Step 3: Find $a^2 + b^2$. We know $|z| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5$. Therefore, $a^2 + b^2 = 5$.
Step 4: Solve for a and b. We have two equations: 1) $a^2 + b^2 = 5$ 2) $a^2 - b^2 = 3$
Add the equations: $2a^2 = 8 \implies a^2 = 4 \implies a = \pm 2$. Subtract the equations: $2b^2 = 2 \implies b^2 = 1 \implies b = \pm 1$.
Step 5: Determine correct signs. Since $2ab = -4$ (which is negative), a and b must have opposite signs. If $a = 2$, then $b = -1$. If $a = -2$, then $b = 1$.
Final Answer: The square roots of $3 - 4i$ are $\pm(2 - i)$.

9.5 De Moivres Theorem

Methodology:

De Moivres Theorem Formula and Application For any real number n:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Useful Corollaries:

1. $(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$

2. $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

3. $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

This theorem is extremely useful for calculating large powers of complex numbers.

1. Convert the complex number to polar form: $z = r(\cos\theta + i\sin\theta)$

2. Apply the power n: $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

Worked Example:

Simplify using De Moivres Theorem Evaluate $(\sqrt{3}+i)^6$.
Step 1: Convert to polar form. Let $z = \sqrt{3}+i$. Modulus $r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$. Argument $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ (Quadrant I). So, $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
Step 2: Apply the power using De Moivre's Theorem.

$$z^6 = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^6$$

$$= 2^6\left(\cos\left(6\cdot\frac{\pi}{6}\right) + i\sin\left(6\cdot\frac{\pi}{6}\right)\right)$$

Step 3: Simplify.

$$= 64(\cos\pi + i\sin\pi)$$

Since $\cos\pi = -1$ and $\sin\pi = 0$:

$$= 64(-1+0) = -64$$

Final Answer: -64

Worked Example:

Simplify fraction of polar expressions Simplify the expression: $\frac{(\cos 3\theta + i\sin 3\theta)^4(\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3(\cos 5\theta + i\sin 5\theta)^{-4}}$.
Step 1: Apply De Moivre's Theorem to simplify powers. Numerator: $(\cos 3\theta + i\sin 3\theta)^4 = \cos(12\theta) + i\sin(12\theta) = e^{i12\theta}$ $(\cos 4\theta - i\sin 4\theta)^5 = (\cos(-4\theta) + i\sin(-4\theta))^5 = \cos(-20\theta) + i\sin(-20\theta) = e^{-i20\theta}$ Denominator: $(\cos 4\theta + i\sin 4\theta)^3 = \cos(12\theta) + i\sin(12\theta) = e^{i12\theta}$ $(\cos 5\theta + i\sin 5\theta)^{-4} = \cos(-20\theta) + i\sin(-20\theta) = e^{-i20\theta}$
Step 2: Substitute and cancel. It's easier to use the exponential notation or simply add/subtract arguments.

$$\text{Expression} = \frac{e^{i12\theta}\cdot e^{-i20\theta}}{e^{i12\theta}\cdot e^{-i20\theta}}$$

Step 3: Simplify. The numerator and denominator are exactly the same:

$$= \frac{e^{-i8\theta}}{e^{-i8\theta}} = 1$$

Final Answer: 1

CHAPTER 10

Complex Numbers

10.1 Standard Form $(a + ib)$ and Algebra

Methodology:

How to express a complex number in standard form $a + ib$ **Goal:** Simplify given complex mathematical expressions into the format $z = a + ib$, where a is the real part and b is the imaginary part.

1. **Addition and Subtraction:** Group the real parts together and the imaginary parts together.
2. **Multiplication:** Expand the brackets just like polynomials. Replace any i^2 with -1.
3. **Division:** To simplify a fraction $\frac{z_1}{z_2}$, multiply both the numerator and the denominator by the **conjugate** of the denominator. Expand, use $i^2 = -1$, and separate into real and imaginary parts.

Worked Example:

Multiplication and Division to Standard Form **Problem:** Express the complex number $z = \frac{(2+3i)(1-2i)}{3+4i}$ in the form $a + ib$.
Solution:
Step 1: Simplify the numerator through multiplication.

$$(2 + 3i)(1 - 2i) = 2(1) + 2(-2i) + 3i(1) + 3i(-2i)$$
$$= 2 - 4i + 3i - 6i^2$$

Since $i^2 = -1$:

$$= 2 - i - 6(-1) = 2 - i + 6 = 8 - i$$

So, the expression becomes $z = \frac{8-i}{3+4i}$.
Step 2: Multiply numerator and denominator by the conjugate of the denominator. The conjugate of the denominator $(3 + 4i)$ is $(3 - 4i)$.

$$z = \frac{8 - i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i}$$

Step 3: Expand both parts and replace i^2 with -1. Numerator:

$$(8 - i)(3 - 4i) = 24 - 32i - 3i + 4i^2 = 24 - 35i + 4(-1) = 20 - 35i$$

Denominator:

$$(3 + 4i)(3 - 4i) = 3^2 - (4i)^2 = 9 - 16i^2 = 9 - 16(-1) = 9 + 16 = 25$$

Step 4: Separate into $a + ib$ form.

$$z = \frac{20 - 35i}{25} = \frac{20}{25} - \frac{35}{25}i = \frac{4}{5} - \frac{7}{5}i$$

Thus, the standard form is $a + ib$ where $a = \frac{4}{5}$ and $b = -\frac{7}{5}$.

10.2 Conjugate, Modulus, and Inverse

Methodology:

How to find the Conjugate, Modulus, and Inverse Given a complex number $z = a + ib$:

1. **Conjugate ($\bar{z}$):** Change the sign of the imaginary part. $\bar{z} = a - ib$.
2. **Modulus ($|z|$):** Use the distance formula. $|z| = \sqrt{a^2 + b^2}$. Note that this is always a positive real number.
3. **Multiplicative Inverse (z^{-1}):** Use the formula $z^{-1} = \frac{\bar{z}}{|z|^2}$ or compute $\frac{1}{a+ib}$ directly by multiplying the numerator and denominator by the conjugate.

Worked Example:

Finding Conjugate, Modulus, and Inverse **Problem:** Find the conjugate, modulus, and multiplicative inverse of $z = 3 - 4i$.
Solution:
Step 1: Identify a and b. Here, $a = 3$ and $b = -4$.
Step 2: Find the Conjugate ($\bar{z}$). Change the sign of the imaginary part:

$$\bar{z} = 3 + 4i$$

Step 3: Find the Modulus ($|z|$).

$$|z| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Step 4: Find the Multiplicative Inverse (z^{-1}). Using the formula $z^{-1} = \frac{\bar{z}}{|z|^2}$:

$$|z|^2 = 5^2 = 25$$

$$z^{-1} = \frac{3 + 4i}{25} = \frac{3}{25} + \frac{4}{25}i$$

10.3 Argument and Polar Form

Methodology:

How to find the Argument and express in Polar Form **Goal:** Convert a complex number $z = a + ib$ into its polar form $z = r(\cos\theta + i\sin\theta)$.

1. **Find the Modulus (r):** Calculate $r = |z| = \sqrt{a^2 + b^2}$.
2. **Find the Principal Argument (θ):**
   - First, calculate the reference angle $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$.
   - Determine the quadrant of (a, b):
     - **Quadrant I ($a > 0, b > 0$):** $\theta = \alpha$
     - **Quadrant II ($a < 0, b > 0$):** $\theta = \pi - \alpha$
     - **Quadrant III ($a < 0, b < 0$):** $\theta = -(\pi - \alpha)$ or $\theta = \alpha - \pi$
     - **Quadrant IV ($a > 0, b < 0$):** $\theta = -\alpha$
3. **Write in Polar Form:** Substitute r and θ into $z = r(\cos\theta + i\sin\theta)$.

Worked Example:

Converting to Polar Form (Quadrant II) **Problem:** Express $z = -1 + \sqrt{3}i$ in polar form.
Solution:
Step 1: Identify a and b. $a = -1$, $b = \sqrt{3}$.
Step 2: Find the Modulus (r).

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

Step 3: Calculate reference angle α.

$$\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

Step 4: Determine the quadrant to find θ. Since a is negative and b is positive, $(-1, \sqrt{3})$ lies in **Quadrant II**.

$$\theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Step 5: Write the polar form.

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

10.4 De Moivre's Theorem

Methodology:

How to use De Moivre's Theorem **Goal:** Evaluate large integer powers of complex numbers, like $(a+ib)^n$. De Moivre's Theorem states: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

1. **Convert to Polar Form:** If the number is given in $a + ib$ format, first find its modulus r and argument θ to express it as $r(\cos\theta + i\sin\theta)$.
2. **Apply the Theorem:** Raise r to the power of n and multiply the angle θ by n.
3. **Simplify and Convert Back (if required):** Evaluate the new trigonometric values $\cos(n\theta)$ and $\sin(n\theta)$ to get the final answer in $a + ib$ form.

Worked Example:

Applying De Moivre's Theorem **Problem:** Simplify $(1+i)^8$ using De Moivre's Theorem.
Solution:
Step 1: Convert $1+i$ to polar form. Here $a = 1, b = 1$. Modulus $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Both a and b are positive (Quadrant I). $\alpha = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Thus, $\theta = \frac{\pi}{4}$. Polar form: $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.
Step 2: Apply De Moivre's Theorem for $n = 8$.

$$(1+i)^8 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^8$$

$$= (\sqrt{2})^8\left(\cos\left(8 \times \frac{\pi}{4}\right) + i\sin\left(8 \times \frac{\pi}{4}\right)\right)$$

Step 3: Simplify the expression. $(\sqrt{2})^8 = (2^{1/2})^8 = 2^4 = 16$. The angle simplifies: $8 \times \frac{\pi}{4} = 2\pi$.

$$= 16(\cos 2\pi + i\sin 2\pi)$$

Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$:

$$= 16(1 + i(0)) = 16$$

10.5 Square Root of a Complex Number

Methodology:

How to find the Square Root of a Complex Number **Goal:** Find $\sqrt{a+ib} = x+iy$.

1. **Set up the equation:** Let $\sqrt{a+ib} = x+iy$.
2. **Square both sides:** $a+ib = (x^2-y^2)+i(2xy)$.
3. **Equate real and imaginary parts:**
   - Equation 1: $x^2-y^2=a$
   - Equation 2: $2xy=b$
4. **Use the algebraic identity:** $(x^2+y^2)^2 = (x^2-y^2)^2+(2xy)^2 = a^2+b^2$.
5. **Solve for x^2+y^2:** Since x, y are real numbers, $x^2+y^2=\sqrt{a^2+b^2}$ (which is the modulus $|z|$).
6. **Find x and y:** Add and subtract the equations $(x^2-y^2=a)$ and $(x^2+y^2=|z|)$ to find x^2 and y^2. Take the square root to find values for x and y.
7. **Determine the signs:** Use the sign of b (from $2xy=b$) to choose matching pairs of (x, y). If $b>0$, x and y have the same sign. If $b<0$, they have opposite signs.

Worked Example:

Finding the Square Root **Problem:** Find the square root of $5-12i$.
Solution:
Step 1: Set up equations. Let $\sqrt{5-12i}=x+iy$. Squaring both sides gives: $5-12i=(x^2-y^2)+i(2xy)$. This gives two equations:

$$x^2-y^2=5 \quad \text{(Eq. 1)}$$
$$2xy=-12 \quad \text{(Eq. 2)}$$

Step 2: Use the identity to find x^2+y^2.

$$(x^2+y^2)=\sqrt{a^2+b^2}=\sqrt{(5)^2+(-12)^2}=\sqrt{25+144}=\sqrt{169}=13$$

So, $x^2+y^2=13 \quad$ (Eq. 3).
Step 3: Solve for x and y. Add (Eq. 1) and (Eq. 3):

$$(x^2-y^2)+(x^2+y^2)=5+13 \implies 2x^2=18 \implies x^2=9 \implies x=\pm 3$$

Subtract (Eq. 1) from (Eq. 3):

$$(x^2+y^2)-(x^2-y^2)=13-5 \implies 2y^2=8 \implies y^2=4 \implies y=\pm 2$$

Step 4: Determine valid (x, y) pairs. From (Eq. 2), $2xy=-12$. Since this product is negative, x and y must have opposite signs. Thus, the valid pairs are $(3, -2)$ and $(-3, 2)$.
The square roots are: $\pm(3-2i)$.

10.6 Cube Roots of Unity

Methodology:

How to solve problems involving Cube Roots of Unity **Goal:** Simplify expressions containing $1, \omega$, and ω^2. The cube roots of unity are 1, $\omega = \frac{-1+\sqrt{3}i}{2}$, and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$.

1. **Use the Sum Property:** $1 + \omega + \omega^2 = 0$. You will frequently use variations of this to substitute terms:
   - $1 + \omega = -\omega^2$
   - $1 + \omega^2 = -\omega$
   - $\omega + \omega^2 = -1$
2. **Use the Product Property:** $\omega^3 = 1$. To simplify higher powers like ω^n, divide n by 3 and find the remainder r. Then $\omega^n = \omega^r$. For example, $\omega^7 = \omega^1 = \omega$.
3. **Substitute and Simplify:** Replace combinations of terms using these two fundamental rules until the expression is fully reduced.

Worked Example:

Evaluating Expressions with Cube Roots of Unity **Problem:** Evaluate $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$.
Solution:
Step 1: Simplify the first term using $1 + \omega^2 = -\omega$. The expression inside the first bracket is $1 - \omega + \omega^2$. Rearrange it to group terms: $(1 + \omega^2) - \omega$. Substitute $(1 + \omega^2)$ with $-\omega$:

$$(-\omega) - \omega = -2\omega$$

So, the first term becomes $(-2\omega)^5 = -32\omega^5$.
Step 2: Simplify the second term using $1 + \omega = -\omega^2$. The expression inside the second bracket is $1 + \omega - \omega^2$. Substitute $(1 + \omega)$ with $-\omega^2$:

$$(-\omega^2) - \omega^2 = -2\omega^2$$

So, the second term becomes $(-2\omega^2)^5 = -32\omega^{10}$.
Step 3: Reduce powers of ω using $\omega^3 = 1$. For ω^5: 5 divided by 3 leaves a remainder of 2, so $\omega^5 = \omega^2$. For ω^{10}: 10 divided by 3 leaves a remainder of 1, so $\omega^{10} = \omega$.
Step 4: Combine the simplified terms.

$$-32\omega^5 + (-32\omega^{10}) = -32\omega^2 - 32\omega$$

Factor out -32:

$$-32(\omega^2 + \omega)$$

From the sum property, $\omega^2 + \omega = -1$.

$$-32(-1) = 32$$

The final answer is 32.

Appendix: Key Formulae

This appendix provides a quick reference to the key formulae and mathematical relationships discussed in this book, categorized by topic.

1. Matrices and Determinants

Determinant of a 2×2 Matrix

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is:

$$|A| = ad - bc$$

Inverse of a Matrix

The inverse of a square matrix A is given by:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

where $\text{adj}(A)$ is the adjoint matrix of A, and $|A| \neq 0$.

Properties of Transpose

$$(A + B)^T = A^T + B^T$$
$$(AB)^T = B^T A^T$$

Solving Systems of Linear Equations

For a system $AX = B$, the solution is given by:

$$X = A^{-1}B$$

2. Complex Numbers

Modulus and Argument

For a complex number $z = x + iy$:

$$\text{Modulus: } |z| = r = \sqrt{x^2 + y^2}$$

$$\text{Argument: } \alpha = \tan^{-1}\left|\frac{y}{x}\right|$$

The principal argument θ depends on the quadrant in which the complex number lies.

Polar Form

$$z = r(\cos\theta + i\sin\theta)$$

De Moivre's Theorem

For any integer n:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Square Root of a Complex Number

To find the square root of $a + ib = x + iy$, we use:

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$$

where $x^2 - y^2 = a$ and $2xy = b$.

Cube Roots of Unity

The cube roots of unity are $1, \omega, \omega^2$.

$$1 + \omega + \omega^2 = 0$$
$$\omega^3 = 1$$

3. Differential Calculus

Product Rule

If $y = u \cdot v$, then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

If $y = \frac{u}{v}$, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Parametric Differentiation

If $x = f(\theta)$ and $y = g(\theta)$, then:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

Logarithmic Differentiation

For functions of the form $y = [u(x)]^{v(x)}$:

$$\log y = v(x) \cdot \log(u(x))$$

Differentiating both sides implicitly yields $\frac{dy}{dx}$.

4. Integral Calculus

Standard Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Properties of Definite Integrals

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

5. Applications of Calculus

Kinematics

If $s(t)$ represents the displacement of a particle at time t:

$$\text{Velocity: } v = \frac{ds}{dt}$$

$$\text{Acceleration: } a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Area Under a Curve

The area A bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$, $x = b$ is:

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

6. Differential Equations

Variable Separable Form

If a differential equation can be written as $f(x)\,dx = g(y)\,dy$, the solution is:

$$\int g(y)\,dy = \int f(x)\,dx + C$$

Linear Differential Equations

A first-order linear differential equation has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The Integrating Factor (I.F.) is:

$$\text{I.F.} = e^{\int P(x)\,dx}$$

The general solution is given by:

$$y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})\,dx + C$$